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The Quantum-Corrected Fermion Mode Function during Inflation

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ABSTRACT

My project computed the one loop fermion self-energy for massless Dirac + Einstein in the presence of a locally de Sitter background. I employed dimensional regularization and obtain a fully renormalized result by absorbing all divergences with Bogliubov, Parasiuk, Hepp and Zimmermann (BPHZ) counterterms. An interesting technical aspect of my computation was the need for a noninvariant counterterm, owing to the breaking of de Sitter invariance by our gauge condition. I also solved the effective Dirac equation for massless fermions during inflation in the simplest gauge, including all one loop corrections from quantum gravity. At late times the result for a spatial plane wave behaves as if the classical solution were subjected to a time-dependent field strength renormalization of $Z_2(t) = 1 - \frac{17}{4\pi} G H^2 \ln(a) + O(G^2)$. I showed that this also follows from making the Hartree approximation, although the numerical coefficients differ.

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To my dearest aunt, Hsiu-Lian Chuang

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1 INTRODUCTION

My research focussed on infer how quantum gravity affects massless fermions at one loop order in the inflationary background geometry which corresponds to a locally de Sitter space. In the following sections, we will discuss what inflation is, why it enhances the effect of quantum gravity, how one can study this enhancement and why reliable conclusions can be reached in spite of the fact that a completely consistent theory of quantum gravity is not yet known.

1.1 Inflation

On the largest scales our universe is amazingly homogeneous and isotropic. It also seems to have nearly zero spatial curvature [1]. Based on these three features our universe can be described by the following geometry,

$$ds^2 = -dt^2 + a^2(t) d\vec{x} \cdot d\vec{x} \ . \tag{1}$$

The coordinate t is physical time. The function $a(t)$ is called the scale factor. This is because it converts Euclidean coordinate distance $\|\vec{x}-\vec{y}\|$ into physical distance $a(t)\|\vec{x}-\vec{y}\|$.

From the scale factor we form the redshift $z(t)$, the Hubble parameter $H(t)$ as well as the deceleration parameter $q(t)$. Their definitions are:

$$z(t) \equiv \frac{a_0}{a(t)} - 1 \quad , \quad H(t) \equiv \frac{\dot{a}}{a} \quad , \quad q(t) \equiv -\frac{a\ddot{a}}{\dot{a}^2} = -1 - \frac{\dot{H}}{H^2} \ . \tag{2}$$

The Hubble parameter $H(t)$ tells us the rate at which the universe is expanding. The deceleration parameter measures the fractional acceleration rate $(\ddot{a}/a)$ in units of Hubble parameter. The current value of Hubble parameter is , $H_0 = (71^{+4}_{-3}) \frac{\mathrm{Km/s}}{\mathrm{Mpc}} \simeq 2.3 \times 10^{-18}\mathrm{Hz}$ [1]. From the observation of Type Ia supernovae one can infer $q_0 \simeq -0.6$ [2], which is consistent with a universe which is currently about 30% matter and 70% vacuum energy.

Inflation is defined as accelerated expansion, that is, $q(t) < 0$ as well as $H(t) > 0$. During the epoch of primordial inflation the Hubble parameter may have been as large as $H_I \sim 10^{37}\mathrm{Hz}$ and the deceleration parameter is thought to have been infinitesimally greater than -1. The current values of the cosmological parameters are consistent with inflation, however, the phenomenological interest in my calculation concerns primordial inflation.

1.2 Uncertainty Principle during Inflation

To understand quantum effects during inflation it is instructive to review the energy-time uncertainty principle,

$$\triangle E \triangle t \gtrsim 1 \; . \tag{3}$$

Consider the process of a pair of virtual particles emerging from the vacuum. This process can conserve 3-momentum if the particles have $\pm\vec{k}$ but it must violate energy conservation. If the particles have mass m then each of them has energy,

$$E(\vec{k}) = \sqrt{m^2 + \parallel \vec{k} \parallel^2} \; . \tag{4}$$

The energy-time uncertainty principle restricts how long a virtual pair of such particles with $\pm\vec{k}$ can exist. If the pair was created at time t, it can last for a time Δt given by the inequality,

$$2E(\vec{k}) \triangle t \lesssim 1 \; . \tag{5}$$

The lifetime of the pair is therefore

$$\triangle t = \frac{1}{2E(\vec{k})} \; . \tag{6}$$

One can see that in flat spacetime all particles with $\vec{k} \neq 0$ have a finite lifetime, and that massless particles live longer than massive particles with the same $\vec{k}$.

How does this change during inflation? Because the homogeneous and isotropic geometry shown by Equation 1 possesses spatial translation invariance it follows that particles are still labeled by constant wave numbers $\vec{k}$, just as in flat space. However, because $\vec{k}$ involves an inverse length, which must be multiplied by the scale factor $a(t)$ to give the physical length, the physical wave number is $\vec{k}/a(t)$. Therefore the physical energy is not Equation 4 but rather,

$$E(t, \vec{k}) = \sqrt{m^2 + \parallel \vec{k} \parallel^2 / a^2(t)} \; . \tag{7}$$

The left-hand side of the previous inequality becomes an integral:

$$\int_t^{t+\triangle t} dt' 2E(t', \vec{k}) \lesssim 1 \tag{8}$$

Obviously anything that reduces $E(t', \vec{k})$ increases $\triangle t$. Therefore let us consider zero mass. Zero mass will simplify the integrand in Equation 8 to $2\|\vec{k}\|/a(t')$. If the scale factor $a(t)$ grows fast enough, the quantity $2\|\vec{k}\|/a(t')$ becomes so small that the integral will be dominated by the lower limit and the inequality of Equation 8 can remain satisfied even though $\triangle t$ goes to infinity. Under these conditions with $m = 0$ and $a(t) = a_I e^{Ht}$, Equation 8 gives,

$$\frac{2 \parallel \vec{k} \parallel}{Ha(t)}(1 - e^{-H\triangle t}) \lesssim 1 \; . \tag{9}$$

From this discussion we conclude that massless virtual particles can live forever during inflation if they emerge with $\parallel \vec{k} \parallel \lesssim Ha(t)$.

1.3 Crucial Role of Conformal Invariance

One might think that the big obstacle to inflationary particle production is nonzero mass. However, the scale of primordial inflation is so high that a lot of particles are effectively massless and they nevertheless experience little inflationary production. The reason is that they possess a symmetry called "conformal invariance."

A simple conformally invariant theory is electromagnetism in $D = 4$ spacetime dimensions. Consider D dimensional electromagnetism,

$$\mathcal{L}_{EM} = -\frac{1}{4} F_{\alpha\beta} F_{\rho\sigma} g^{\alpha\rho} g^{\beta\sigma} \sqrt{-g} \; , \tag{10}$$

where $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$. Under a conformal transformation $g'_{\mu\nu} = \Omega^2(x) g_{\mu\nu}$ and $A'_\mu = A_\mu$ the Lagrangian becomes,

$$\mathcal{L}' = F_{\alpha\beta} F_{\rho\sigma} \Omega^{-2} g^{\alpha\rho} \Omega^{-2} g^{\beta\sigma} \Omega^D \sqrt{-g} = \mathcal{L} \Omega^{D-4} \tag{11}$$

Hence electromagnetism is conformally invariant in $D = 4$. Other conformally invariant theories are the massless conformally coupled scalar,

$$\mathcal{L} = -\frac{1}{2} \partial_\mu \phi \partial_\nu \phi g^{\mu\nu} \sqrt{-g} - \frac{1}{8} \Bigl(\frac{D-2}{D-1}\Bigr) \phi^2 R \sqrt{-g} \; . \tag{12}$$

and massless fermions,

$$\mathcal{L} \;\; = \;\; \overline{\psi} e^\mu{}_b \gamma^b (i\partial_\mu - \frac{1}{2} A_{\mu cd} J^{cd}) \psi \sqrt{-g} \, . \tag{13}$$

Here $\phi' = \Omega^{1-\frac{D}{2}}\phi$ and $\psi' = \Omega^{\frac{1-D}{2}}\psi$ under a conformal transformation.

If the theory possesses conformal invariance, it is much more convenient to express the homogeneous and isotropic geometry of Equation 1 in conformal coordinates,

$$\begin{aligned} dt = a(t)d\eta \Longrightarrow ds^2 &= -dt^2 + a^2(t)d\vec{x}\cdot d\vec{x} \\ &= a^2(t)(-d\eta^2 + d\vec{x}\cdot d\vec{x}\,)\;. \end{aligned} \tag{14}$$

Here t is physical time and η is conformal time. In the $(\eta, \vec{x})$ coordinates, conformally invariant theories are locally identical to their flat space cousins. The rate at which virtual particles emerge from the vacuum per unit conformal time must be the same constant — call it Γ — as in flat space. Hence the rate of emergence per unit physical time is,

$$\frac{dN}{dt} = \frac{dN}{d\eta}\frac{d\eta}{dt} = \frac{\Gamma}{a(t)}\;. \tag{15}$$

One can see that the emergence rate in a locally de Sitter background is suppressed by a factor of $1/a$ ($a \sim e^{Ht}$, $H > 0$). Therefore any conformally invariant, massless virtual particles with $\parallel \vec{k} \parallel \lesssim Ha(t)$ can live forever but the problem is that they don't have much chance to emerge from the vacuum.

1.4 Gravitons and Massless Minimally Coupled Scalars

Not every massless particle is conformally invariant. Two exceptions are gravity and the massless minimally coupled (MMC) scalar,

$$\mathcal{L} = \frac{1}{16\pi G}(R - 2\Lambda)\sqrt{-g}\;, \tag{16}$$

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial_\nu\phi g^{\mu\nu}\sqrt{-g}\;. \tag{17}$$

Here R is the Ricci scalar and Λ is the cosmological constant. From previous sections one can conclude that big quantum effects come from combining

- Inflation;
- Massless particles; and
- The absence of invariance.

Therefore one can conclude that gravitons and MMC scalars have the potential to mediate vastly enhanced quantum effects during inflation because they are simultaneously massless and not conformally invariant.

To see that the production of gravitons and MMC scalars is not suppressed during inflation note that each polarization and wave number behaves like a harmonic oscillator [3, 4],

$$L = \frac{1}{2} m \dot{q}^2 - \frac{1}{2} m \omega^2 q^2 \; , \tag{18}$$

with time dependent mass $m(t) = a^3(t)$ and frequency $\omega(t) = \frac{k}{a(t)}$. The Heisenberg equation of motion can be solved in terms of mode functions $u(t,k)$ and canonically normalized raising and lowering operators $\alpha^\dagger$ and α,

$$\ddot{q} + 3H\dot{q} + \frac{k^2}{a^2} q = 0 \quad \Longrightarrow \quad q(t) = u(t,k)\alpha + u^*(t,k)\alpha^\dagger \quad \text{with} \quad [\alpha, \alpha^\dagger] = 1 \; , \tag{19}$$

The mode functions $u(t,k)$ are quite complicated for a general scale factor $a(t)$ [5] but they take a simple form for de Sitter,

$$u(t,k) = \frac{H}{\sqrt{2k^3}} \Big[1 - \frac{ik}{Ha(t)} \Big] \exp\Big[\frac{ik}{Ha(t)} \Big] \; . \tag{20}$$

The (co-moving) energy operator for this system is,

$$E(t) = \frac{1}{2} m(t) \dot{q}^2(t) + \frac{1}{2} m(t) \omega^2(t) q^2(t) \; . \tag{21}$$

Owing to the time dependent mass and frequency, there are no stationary states for this system. At any given time the minimum eigenstate of $E(t)$ has energy $\frac{1}{2}\omega(t)$, but which the state changes for each value of time. The state $|\Omega\rangle$ which is annihilated by α has minimum energy in the distant past. The expectation value of the energy operator in this state is,

$$\Big\langle \Omega \Big| E(t) \Big| \Omega \Big\rangle = \frac{1}{2} a^3(t) |\dot{u}(t,k)|^2 + \frac{1}{2} a(t) k^2 |u(t,k)|^2 \Big|_{\text{de Sitter}} = \frac{k}{2a} + \frac{H^2 a}{4k} \; . \tag{22}$$

If one thinks of each particle having energy $k/a(t)$, it follows that the number of particles with any polarization and wave number k grows as the square of the inflationary scale factor,

$$N(t,k) = \Big(\frac{Ha(t)}{2k} \Big)^2 \; . \tag{23}$$

Quantum field theoretic effects are driven by essentially classical physics operating in response to the source of virtual particles implied by quantization. On the basis of Equation 23 one might expect inflation to dramatically enhance quantum effects from MMC scalars and gravitons, and explicit studies over a quarter century have confirmed this. The oldest results are of course the cosmological perturbations induced by scalar inflatons [6] and by gravitons [7]. More recently it was shown that the one-loop vacuum polarization induced by a charged MMC scalar in de Sitter background causes super-horizon photons to behave like massive particles in some ways [8, 9, 10]. Another recent result is that the one-loop fermion self-energy induced by a MMC Yukawa scalar in de Sitter background reflects the generation of a nonzero fermion mass [11, 12].

1.5 Overview

One naturally wonders how interactions with these quanta affect themselves and other particles. The first step in answering this question on the linearized level is to compute the one particle irreducible (1PI) 2-point function for the field whose behavior is in question. This has been done at one loop order for gravitons in pure quantum gravity [13], for photons [8, 9] and charged scalars [14] in scalar quantum electrodynamics (SQED), for fermions [11, 12] and Yukawa scalars [15] in Yukawa theory, for fermions in Dirac + Einstein [16] and, at two loop order, for scalars in ϕ^4 theory [17].

In the first part of my dissertation we compute and renormalize the one loop quantum gravitational corrections to the self-energy of massless fermions in a locally de Sitter background. The physical motivation for this exercise is to check for graviton analogues of the enhanced quantum effects seen in this background for interactions which involve one or more undifferentiated, massless, minimally coupled (MMC) scalars. Those effects are driven by the fact that inflation tends to rip virtual, long wavelength scalars out of the vacuum and thereby lengthens the time during which they can interact with themselves or other particles. Gravitons possess the same crucial property of masslessness without classical conformal invariance that is responsible for the inflationary production of MMC scalars. One might therefore expect a corresponding strengthening of quantum gravitational effects during inflation.

Of particular interest to us is what happens when a MMC scalar is Yukawa coupled to a massless Dirac fermion for non-dynamical gravity. The one loop

fermion self-energy has been computed for this model and used to solve the quantum-corrected Dirac equation [11],

$$\sqrt{-g}\, i\bar{\partial}\!\!\!/_{ij}\psi_j(x) - \int d^4x' \left[{}_i\Sigma_j\right](x;x')\,\psi_j(x') = 0\ . \tag{24}$$

Powers of the inflationary scale factor $a = e^{Ht}$ play a crucial role in understanding this equation for the Yukawa model and also for what we expect from quantum gravity. The Yukawa result for the self-energy [11] consists of terms which were originally ultraviolet divergent and which end up, after renormalization, carrying the same number of scale factors as the classical term. Had the scalar been conformally coupled these would be the only contributions to the one loop self-energy. However, minimally coupled scalars also give contributions due to inflationary particle production. These are ultraviolet finite from the beginning and possesses an extra factor of $a\ln(a)$ relative to the classical term. Higher loops can bring more factors of $\ln(a)$, but no more powers of a, so it is consistent to solve the equation with only the one loop corrections. The result is a drop in wave function which is consistent with the fermion developing a mass that grows as $\ln(a)$. A recent one loop computation of the Yukawa scalar self-mass-squared indicates that the scalar which catalyzes this process cannot develop a large enough mass quickly enough to inhibit the process [15].

Analogous graviton effects should be suppressed by the fact that the $h_{\mu\nu}\overline{\psi}\psi$ interaction of Dirac + Einstein carries a derivative, as opposed to the undifferentiated $\phi\bar{\psi}\psi$ interaction of Yukawa theory. What we expect is that the corresponding quantum gravitational self-energy will consist of two terms. The most ultraviolet singular one will require higher derivative counterterms and will end up, after renormalization, possessing *one less* factor of a than the classical term. The less singular term due to inflationary particle production should require only lower derivative counterterms and will be enhanced from the classical term by a factor of $\ln(a)$. This would give a much weaker effect than the analogous term in the Yukawa model, but it would still be interesting. And note that any such effect from gravitons would be universal, independent of assumptions about the existence or couplings of unnaturally light scalars.

The second part of my dissertation consists of using the 1PI 2-point function to correct the linearized equation of motion from Equation 24 for the field in question. We employ the Schwinger-Keldysh formalism to solve for the loop corrected fermion mode function. In the late time limit we find that

the one loop corrected, spatial plane mode functions behave as if the tree order mode functions were simply subject to a time-dependent field strength renormalization. The same result pertains for the Hartree approximation in which the expectation value of the quantum Dirac equation is taken in free graviton vacuum.

1.6 The Issue of Nonrenormalizability

Dirac + Einstein is not perturbatively renormalizable [18], however, ultraviolet divergences can always be absorbed in the BPHZ sense [19, 20, 21, 22]. A widespread misconception exists that no valid quantum predictions can be extracted from such an exercise. This is false: while nonrenormalizability does preclude being able to compute *everything*, that not the same thing as being able to compute *nothing*. The problem with a nonrenormalizable theory is that no physical principle fixes the finite parts of the escalating series of BPHZ counterterms needed to absorb ultraviolet divergences, order-by-order in perturbation theory. Hence any prediction of the theory that can be changed by adjusting the finite parts of these counterterms is essentially arbitrary. However, loops of massless particles make nonlocal contributions to the effective action that can never be affected by local counterterms. These nonlocal contributions typically dominate the infrared. Further, they cannot be affected by whatever modification of ultraviolet physics ultimately results in a completely consistent formalism. As long as the eventual fix introduces no new massless particles, and does not disturb the low energy couplings of the existing ones, the far infrared predictions of a BPHZ-renormalized quantum theory will agree with those of its fully consistent descendant.

It is worthwhile to review the vast body of distinguished work that has exploited this fact. The oldest example is the solution of the infrared problem in quantum electrodynamics by Bloch and Nordsieck [23], long before that theory's renormalizability was suspected. Weinberg [24] was able to achieve a similar resolution for quantum gravity with zero cosmological constant. The same principle was at work in the Fermi theory computation of the long range force due to loops of massless neutrinos by Feinberg and Sucher [25, 26]. Matter which is not supersymmetric generates nonrenormalizable corrections to the graviton propagator at one loop, but this did not prevent the computation of photon, massless neutrino and massless, conformally coupled scalar loop corrections to the long range gravitational force [27, 28, 29, 30]. More recently, Donoghue [31, 32] has touched off a minor

industry [33, 34, 35, 36, 37] by applying the principles of low energy effective field theory to compute graviton corrections to the long range gravitational force. Our analysis exploits the power of low energy effective field theory in the same way, differing from the previous examples only in the detail that our background geometry is locally de Sitter rather than flat.[1]

2 FEYNMAN RULES

When the geometry is Minkowski, we work in momentum space because of spacetime translation invariance. This symmetry is broken in de Sitter background so propagators and vertices are no longer simple in momentum space. Therefore we require Feynman rules in position space. We start from the general Dirac Lagrangian which is conformally invariant. We exploit this by conformally rescaling the fields to obtain simple expressions for the fermion propagator and the vertex operators. However, there are several subtleties for the graviton propagator. First of all, the Einstein theory is not conformally invariant. Secondly, there is a poorly understood obstacle to adding a de Sitter invariant gauge-fixing term to the action. We avoid this by adding a gauge-fixing term which breaks de Sitter invariance. That gives correct physics but it leads to the third problem, which is the possibility of noninvariant counterterms. Fortunately, only one of these occurs.

2.1 Fermions in Quantum Gravity

The coupling of gravity to particles with half integer spin is usually accomplished by shifting the fundamental gravitational field variable from the metric $g_{\mu\nu}(x)$ to the vierbein $e_{\mu m}(x)$.[2] Greek letters stand for coordinate indices and Latin letters denote Lorentz indices, and both kinds of indices take values in the set $\{0, 1, 2, \ldots, (D-1)\}$. One recovers the metric by contracting two vierbeins into the Lorentz metric η^{bc},

$$g_{\mu\nu}(x) = e_{\mu b}(x) e_{\nu c}(x) \eta^{bc} \; . \tag{25}$$

The coordinate index is raised and lowered with the metric ($e^{\mu}{}_{b} = g^{\mu\nu} e_{\nu b}$), while the Lorentz index is raised and lowered with the Lorentz metric ($e_{\mu}{}^{b} =$

[1]For another recent example in a nontrivial cosmology see D. Espriu, T. Multamäki and E. C. Vagenas, Phys. Lett. B628 (2005) 197, gr-qc/0503033.

[2] For another approach see H. A. Weldon, Phys. Rev. D63 (2001) 104010, gr-qc/0009086.

$\eta^{bc} e_{\mu c}$). We employ the usual metric-compatible and vierbein-compatible connections,

$$g_{\rho\sigma;\mu} = 0 \implies \Gamma^{\rho}{}_{\mu\nu} = \frac{1}{2} g^{\rho\sigma} \Bigl(g_{\sigma\mu,\nu} + g_{\nu\sigma,\mu} - g_{\mu\nu,\sigma} \Bigr) \; , \tag{26}$$

$$e_{\beta b;\mu} = 0 \implies A_{\mu cd} = e^{\nu}{}_{c} \Bigl(e_{\nu d,\mu} - \Gamma^{\rho}{}_{\mu\nu} e_{\rho d} \Bigr) \; . \tag{27}$$

Fermions also require gamma matrices, γ^b_{ij}. The anti-commutation relations,

$$\Bigl\{ \gamma^b , \gamma^c \Bigr\} \equiv \Bigl(\gamma^b \gamma^c + \gamma^c \gamma^b \Bigr) = -2 \eta^{bc} I \; , \tag{28}$$

imply that only fully anti-symmetric products of gamma matrices are actually independent. The Dirac Lorentz representation matrices are such an anti-symmetric product,

$$J^{bc} \equiv \frac{i}{4} \Bigl(\gamma^b \gamma^c - \gamma^c \gamma^b \Bigr) \equiv \frac{i}{2} \gamma^{[b} \gamma^{c]} \; . \tag{29}$$

They can be combined with the spin connection of Equation 27 to form the Dirac covariant derivative operator,

$$\mathcal{D}_\mu \equiv \partial_\mu + \frac{i}{2} A_{\mu cd} J^{cd} \; . \tag{30}$$

Other identities we shall often employ involve anti-symmetric products,

$$\gamma^b \gamma^c \gamma^d = \gamma^{[b} \gamma^c \gamma^{d]} - \eta^{bc} \gamma^d + \eta^{db} \gamma^c - \eta^{cd} \gamma^b \; , \tag{31}$$

$$\gamma^b J^{cd} = \frac{i}{2} \gamma^{[b} \gamma^c \gamma^{d]} + \frac{i}{2} \eta^{bd} \gamma^c - \frac{i}{2} \eta^{bc} \gamma^d \; . \tag{32}$$

We shall also encounter cases in which one gamma matrix is contracted into another through some other combination of gamma matrices,

$$\gamma^b \gamma_b = -D I \; , \tag{33}$$

$$\gamma^b \gamma^c \gamma_b = (D-2) \gamma^c \; , \tag{34}$$

$$\gamma^b \gamma^c \gamma^d \gamma_b = 4 \eta^{cd} I - (D-4) \gamma^c \gamma^d \; , \tag{35}$$

$$\gamma^b \gamma^c \gamma^d \gamma^e \gamma_b = 2 \gamma^e \gamma^d \gamma^c + (D-4) \gamma^c \gamma^d \gamma^e \; . \tag{36}$$

The Lagrangian of massless fermions is,

$$\mathcal{L}_{\rm Dirac} \equiv \overline{\psi} e^{\mu}{}_{b} \gamma^b i \mathcal{D}_\mu \psi \sqrt{-g} \; . \tag{37}$$

Because our locally de Sitter background is conformally flat it is useful to rescale the vierbein by an arbitrary function of spacetime $a(x)$,

$$e_{\beta b} \equiv a\,\widetilde{e}_{\beta b} \qquad \Longrightarrow \qquad e^{\beta b} = a^{-1}\,\widetilde{e}^{\beta b} \; . \tag{38}$$

Of course this implies a rescaled metric $\widetilde{g}_{\mu\nu}$,

$$g_{\mu\nu} = a^2\,\widetilde{g}_{\mu\nu} \qquad \Longrightarrow \qquad g^{\mu\nu} = a^{-2}\,\widetilde{g}^{\mu\nu} \; . \tag{39}$$

The old connections can be expressed as follows in terms of the ones formed from the rescaled fields,

$$\Gamma^{\rho}_{\ \ \mu\nu} = a^{-1}\Bigl(\delta^{\rho}_{\ \mu}\,a_{,\nu} + \delta^{\rho}_{\ \nu}\,a_{,\mu} - \widetilde{g}^{\rho\sigma}\,a_{,\sigma}\,\widetilde{g}_{\mu\nu}\Bigr) + \widetilde{\Gamma}^{\rho}_{\ \ \mu\nu} \tag{40}$$

$$A_{\mu cd} = -a^{-1}\Bigl(\widetilde{e}^{\nu}_{\ c}\,\widetilde{e}_{\mu d} - \widetilde{e}^{\nu}_{\ d}\,\widetilde{e}_{\mu c}\Bigr)a_{,\nu} + \widetilde{A}_{\mu cd} \; . \tag{41}$$

We define rescaled fermion fields as follows,

$$\Psi \equiv a^{\frac{D-1}{2}}\psi \qquad \text{and} \qquad \overline{\Psi} \equiv a^{\frac{D-1}{2}}\overline{\psi} \; . \tag{42}$$

The utility of these definitions stems from the conformal invariance of the Dirac Lagrangian,

$$\mathcal{L}_{\text{Dirac}} = \overline{\Psi}\,\widetilde{e}^{\mu}_{\ b}\,\gamma^b\,i\widetilde{\mathcal{D}}_{\mu}\Psi\sqrt{-\widetilde{g}} \; , \tag{43}$$

where $\widetilde{\mathcal{D}}_{\mu} \equiv \partial_{\mu} + \frac{i}{2}\widetilde{A}_{\mu cd}J^{cd}$.

One could follow early computations about flat space background [38, 39] in defining the graviton field as a first order perturbation of the (conformally rescaled) vierbein. However, so much of gravity involves the vierbein only through the metric that it is simpler to instead take the graviton field to be a first order perturbation of the conformally rescaled metric,

$$\widetilde{g}_{\mu\nu} \equiv \eta_{\mu\nu} + \kappa h_{\mu\nu} \qquad \text{with} \qquad \kappa^2 = 16\pi G \; . \tag{44}$$

We then impose symmetric gauge ($e_{\beta b} = e_{b\beta}$) to fix the local Lorentz gauge freedom, and solve for the vierbein in terms of the graviton,

$$\widetilde{e}[\widetilde{g}]_{\beta b} \equiv \Bigl(\sqrt{\widetilde{g}\eta^{-1}}\,\Bigr)_{\beta}^{\ \gamma}\eta_{\gamma b} = \eta_{\beta b} + \frac{1}{2}\kappa h_{\beta b} - \frac{1}{8}\kappa^2 h_{\beta}^{\ \gamma} h_{\gamma b} + \dots \tag{45}$$

It can be shown that the local Lorentz ghosts decouple in this gauge and one can treat the model, at least perturbatively, as if the fundamental variable were the metric and the only symmetry were diffeomorphism invariance [40].

At this stage there is no more point in distinguishing between Latin letters for local Lorentz indices and Greek letters for vector indices. Other conventions are that graviton indices are raised and lowered with the Lorentz metric ($h^{\mu}{}_{\nu} \equiv \eta^{\mu\rho} h_{\rho\nu}$, $h^{\mu\nu} \equiv \eta^{\mu\rho}\eta^{\nu\sigma} h_{\rho\sigma}$) and that the trace of the graviton field is $h \equiv \eta^{\mu\nu} h_{\mu\nu}$. We also employ the usual Dirac "slash" notation,

$$\not{V}_{ij} \equiv V_{\mu} \gamma^{\mu}_{ij} \; . \tag{46}$$

It is straightforward to expand all familiar operators in powers of the graviton field,

$$\widetilde{e}^{\mu}{}_{b} = \delta^{\mu}{}_{b} - \frac{1}{2}\kappa h^{\mu}{}_{b} + \frac{3}{8}\kappa^2 h^{\mu\rho} h_{\rho b} + \dots \; , \tag{47}$$

$$\widetilde{g}^{\mu\nu} = \eta^{\mu\nu} - \kappa h^{\mu\nu} + \kappa^2 h^{\mu}{}_{\rho} h^{\rho\nu} - \dots \; , \tag{48}$$

$$\sqrt{-\widetilde{g}} = 1 + \frac{1}{2}\kappa h + \frac{1}{8}\kappa^2 h^2 - \frac{1}{4}\kappa^2 h^{\rho\sigma} h_{\rho\sigma} + \dots \tag{49}$$

Applying these identities to the conformally rescaled Dirac Lagrangian gives,

$$\begin{aligned} \mathcal{L}_{\rm Dirac} = \overline{\Psi} i \not{\partial} \Psi + \frac{\kappa}{2}\Big\{ h \overline{\Psi} i \not{\partial} \Psi - h^{\mu\nu} \overline{\Psi} \gamma_{\mu} i \partial_{\nu} \Psi - h_{\mu\rho,\sigma} \overline{\Psi} \gamma^{\mu} J^{\rho\sigma} \Psi \Big\} \\ + \kappa^2 \Big\{ \Big[\frac{1}{8} h^2 - \frac{1}{4} h^{\rho\sigma} h_{\rho\sigma}\Big] \overline{\Psi} i \not{\partial} \Psi + \Big[-\frac{1}{4} h h^{\mu\nu} + \frac{3}{8} h^{\mu\rho} h_{\rho}{}^{\nu}\Big] \overline{\Psi} \gamma_{\mu} i \partial_{\nu} \Psi + \Big[-\frac{1}{4} h h_{\mu\rho,\sigma} \\ + \frac{1}{8} h^{\nu}{}_{\rho} h_{\nu\sigma,\mu} + \frac{1}{4} (h^{\nu}{}_{\mu} h_{\nu\rho})_{,\sigma} + \frac{1}{4} h^{\nu}{}_{\sigma} h_{\mu\rho,\nu} \Big] \overline{\Psi} \gamma^{\mu} J^{\rho\sigma} \Psi \Big\} + O(\kappa^3) \; . \end{aligned} \tag{50}$$

From the first term we see that the rescaled fermion propagator is the same as for flat space,

$$i\Big[{}_{i}S_{j}\Big](x;x') = \frac{\Gamma(\frac{D}{2}-1)}{4\pi^{\frac{D}{2}}} \, i \not{\partial}_{ij} \Big(\frac{1}{\Delta x^2}\Big)^{\frac{D}{2}-1} , \tag{51}$$

where the coordinate interval is $\Delta x^2(x;x') \equiv \|\vec{x}-\vec{x}'\|^2 - (|\eta-\eta'| - i\delta)^2$.

We now represent the various interaction terms in Equation 50 as vertex operators acting on the fields. At order κ the interactions involve fields, $\overline{\Psi}_i$, Ψ_j and $h_{\alpha\beta}$, which we number "1", "2" and "3", respectively. Each of the three interactions can be written as some combination $V^{\alpha\beta}_{Iij}$ of tensors, spinors and a derivative operator acting on these fields. For example, the first interaction is,

$$\frac{\kappa}{2} h \overline{\Psi} i \not{\partial} \Psi = \frac{\kappa}{2} \eta^{\alpha\beta} i \not{\partial}_{2ij} \times \overline{\Psi}_i \Psi_j h_{\alpha\beta} \equiv V^{\alpha\beta}_{1ij} \times \overline{\Psi}_i \Psi_j h_{\alpha\beta} \; . \tag{52}$$

Table 1: Vertex operators $U^{\alpha\beta\rho\sigma}_{Iij}$ contracted into $\overline{\Psi}_i \Psi_j h_{\alpha\beta} h_{\rho\sigma}$.

#	Vertex Operator	#	Vertex Operator
1	$\frac{1}{8}\kappa^2 \eta^{\alpha\beta}\eta^{\rho\sigma} i\partial\!\!\!/_{2ij}$	5	$-\frac{1}{4}\kappa^2 \eta^{\alpha\beta}(\gamma^\rho J^{\sigma\mu})_{ij}\partial_{4\mu}$
2	$-\frac{1}{4}\kappa^2 \eta^{\alpha\rho}\eta^{\sigma\beta} i\partial\!\!\!/_{2ij}$	6	$\frac{1}{8}\kappa^2 \eta^{\alpha\rho}(\gamma^\mu J^{\beta\sigma})_{ij}\partial_{4\mu}$
3	$-\frac{1}{4}\kappa^2 \eta^{\alpha\beta}\gamma^\rho_{ij} i\partial^\sigma_2$	7	$\frac{1}{4}\kappa^2 \eta^{\alpha\rho}(\gamma^\beta J^{\sigma\mu})_{ij}(\partial_3 + \partial_4)_\mu$
4	$\frac{3}{8}\kappa^2 \eta^{\alpha\rho}\gamma^\beta_{ij} i\partial^\sigma_2$	8	$\frac{1}{4}\kappa^2 (\gamma^\rho J^{\sigma\alpha})_{ij}\partial^\beta_4$

Hence the 3-point vertex operators are,

$$V^{\alpha\beta}_{1ij} = \frac{\kappa}{2}\eta^{\alpha\beta} i\partial\!\!\!/_{2ij} \quad , \quad V^{\alpha\beta}_{2ij} = -\frac{\kappa}{2}\gamma^{(\alpha}_{ij} i\partial^{\beta)}_2 \quad , \quad V^{\alpha\beta}_{3ij} = -\frac{\kappa}{2}\Bigl(\gamma^{(\alpha} J^{\beta)\mu}\Bigr)_{ij}\partial_{3\mu} \; . \tag{53}$$

The order κ^2 interactions define 4-point vertex operators $U^{\alpha\beta\rho\sigma}_{Iij}$ similarly, for example,

$$\frac{1}{8}\kappa^2 h^2 \overline{\Psi} i\partial\!\!\!/ \Psi = \frac{1}{8}\kappa^2 \eta^{\alpha\beta}\eta^{\rho\sigma} i\partial\!\!\!/_{2ij} \times \overline{\Psi}_i \Psi_j h_{\alpha\beta} h_{\rho\sigma} \equiv U^{\alpha\beta\rho\sigma}_{1ij} \times \overline{\Psi}_i \Psi_j h_{\alpha\beta} h_{\rho\sigma} \; . \tag{54}$$

The eight 4-point vertex operators are given in Table 1. Note that we do not bother to symmetrize upon the identical graviton fields.

2.2 The Graviton Propagator

The gravitational Lagrangian of low energy effective field theory is,

$$\mathcal{L}_{\rm Einstein} \equiv \frac{1}{16\pi G}\Bigl(R - (D\!-\!2)\Lambda\Bigr)\sqrt{-g} \; . \tag{55}$$

The symbols G and Λ stand for Newton's constant and the cosmological constant, respectively. The unfamiliar factor of $D\!-\!2$ multiplying Λ makes the pure gravity field equations imply $R_{\mu\nu} = \Lambda g_{\mu\nu}$ in any dimension. The symbol R stands for the Ricci scalar where our metric is spacelike and our curvature convention is,

$$R \equiv g^{\mu\nu} R_{\mu\nu} \equiv g^{\mu\nu}\Bigl(\Gamma^\rho{}_{\nu\mu,\rho} - \Gamma^\rho{}_{\rho\mu,\nu} + \Gamma^\rho{}_{\rho\sigma}\Gamma^\sigma{}_{\nu\mu} - \Gamma^\rho{}_{\nu\sigma}\Gamma^\sigma{}_{\rho\mu}\Bigr). \tag{56}$$

Unlike massless fermions, gravity is not conformally invariant. However, it is still useful to express it in terms of the rescaled metric of Equation 39 and

connection of Equation 40,

$$\mathcal{L}_{\rm Einstein} = \frac{1}{16\pi G}\Biggl\{a^{D-2}\widetilde{R} - 2(D-1)a^{D-3}\widetilde{g}^{\mu\nu}\Bigl(a_{,\mu\nu} - \widetilde{\Gamma}^{\rho}{}_{\mu\nu}a_{,\rho}\Bigr)$$
$$- (D-4)(D-1)a^{D-4}\widetilde{g}^{\mu\nu}a_{,\mu}a_{,\nu} - (D-2)\Lambda a^D\Biggr\}\sqrt{-\widetilde{g}} \; . \quad (57)$$

The factors of a which complicate this expression are the ultimate reason there is interesting physics in this model!

None of the fermionic Feynman rules depended upon the functional form of the scale factor a because the Dirac Lagrangian is conformally invariant. However, we shall need to fix a in order to work out the graviton propagator from the Einstein Lagrangian in Equation 57. The unique, maximally symmetric solution for positive Λ is known as de Sitter space. In order to regard this as a paradigm for inflation we work on a portion of the full de Sitter manifold known as the open conformal coordinate patch. The invariant element for this is,

$$ds^2 = a^2\Bigl(-d\eta^2 + d\vec{x}\cdot d\vec{x}\Bigr) \qquad \text{where} \qquad a(\eta) = -\frac{1}{H\eta} \; , \quad (58)$$

and the D-dimensional Hubble constant is $H \equiv \sqrt{\Lambda/(D-1)}$. Note that the conformal time η runs from $-\infty$ to zero. For this choice of scale factor we can extract a surface term from the invariant Lagrangian and write it in the form [41],

$$\mathcal{L}_{\rm Einstein} - {\rm Surface} = (\tfrac{D}{2}-1)Ha^{D-1}\sqrt{-\widetilde{g}}\widetilde{g}^{\rho\sigma}\widetilde{g}^{\mu\nu}h_{\rho\sigma,\mu}h_{\nu 0} + a^{D-2}\sqrt{-\widetilde{g}}\widetilde{g}^{\alpha\beta}\widetilde{g}^{\rho\sigma}\widetilde{g}^{\mu\nu}$$
$$\times\Bigl\{\tfrac{1}{2}h_{\alpha\rho,\mu}h_{\beta\sigma,\nu} - \tfrac{1}{2}h_{\alpha\beta,\rho}h_{\sigma\mu,\nu} + \tfrac{1}{4}h_{\alpha\beta,\rho}h_{\mu\nu,\sigma} - \tfrac{1}{4}h_{\alpha\rho,\mu}h_{\beta\sigma,\nu}\Bigr\}. \quad (59)$$

Gauge fixing is accomplished as usual by adding a gauge fixing term. However, it turns out not to be possible to employ a de Sitter invariant gauge for reasons that are not yet completely understood. One can add such a gauge fixing term and then use the well-known formalism of Allen and Jacobson [42] to solve for a fully de Sitter invariant propagator [43, 46, 44, 45, 47]. However, a curious thing happens when one uses the imaginary part of any such propagator to infer what ought to be the retarded Green's function of classical general relativity on a de Sitter background. The resulting Green's function gives a divergent response for a point mass which also fails to obey

the linearized invariant Einstein equation [46]! We stress that the various propagators really do solve the gauge-fixed, linearized equations with a point source. It is the physics which is wrong, not the math. There must be some obstacle to adding a de Sitter invariant gauge fixing term in gravity.

The problem seems to be related to combining constraint equations with the causal structure of the de Sitter geometry. Before gauge fixing the constraint equations are elliptic, and they typically generate a nonzero response throughout the de Sitter manifold, even in regions which are not future-related to the source. Imposing a de Sitter invariant gauge results in hyperbolic equations for which the response is zero in any region that is not future-related to the source. This feature of gauge theories on de Sitter space was first noted by Penrose in 1963 [48] and has since been studied for gravity [41] and electromagnetism [49].

One consequence of the causality obstacle is that no completely de Sitter invariant gauge field propagator can correctly describe even classical physics over the entire de Sitter manifold. The confusing point is the extent of the region over which the original, gauge invariant field equations are violated. For electromagnetism it turns out that a de Sitter invariant gauge can respect the gauge invariant equations on the submanifold which is future-directed from the source [50]. For gravity there seem to be violations of the Einstein equations everywhere [46]. The reason for this difference is not understood.

Quantum corrections bring new problems when using de Sitter invariant gauges. The one loop scalar self-mass-squared has recently been computed in two different gauges for scalar quantum electrodynamics [14]. With each gauge the computation was made for charged scalars which are massless, minimally coupled and for charged scalars which are massless, conformally coupled. What goes wrong is clearest for the conformally coupled scalar, which should experience no large de Sitter enhancement over the flat space result on account of the conformal flatness of the de Sitter geometry. This is indeed the case when one employs the de Sitter breaking gauge that takes maximum account of the conformal invariance of electromagnetism in $D = 3+1$ spacetime dimensions. However, when the computation was done in the de Sitter invariant analogue of Feynman gauge the result was on-shell singularities! Off shell one-particle-irreducible functions need not agree in different gauges [51] but they should agree on shell [52]. In view of its on-shell singularities the result in the de Sitter invariant gauge is clearly wrong.

The nature of the problem may be the apparent inconsistency between de Sitter invariance and the manifold's linearization instability. Any propagator

gives the response (with a certain boundary condition) to a single point source. If the propagator is also de Sitter invariant then this response must be valid throughout the full de Sitter manifold. But the linearization instability precludes solving the invariant field equations for a single point source on the full manifold! This feature of the invariant theory is lost when a de Sitter invariant gauge fixing term is simply added to the action so it must be that the process of adding it was not legitimate. In striving to attain a propagator which is valid everywhere, one invariably obtains a propagator that is not valid anywhere!

Although the pathology has not be identified as well as we should like, the procedure for dealing with it does seem to be clear. One can avoid the problem either by working on the full manifold with a noncovariant gauge condition that preserves the elliptic character of the constraint equations, or else by employing a covariant, but not de Sitter invariant gauge on an open submanifold [41]. We choose the latter course and employ the following analogue of the de Donder gauge fixing term of flat space,

$$\mathcal{L}_{GF} = -\frac{1}{2} a^{D-2} \eta^{\mu\nu} F_\mu F_\nu \; , \; F_\mu \equiv \eta^{\rho\sigma} \Big(h_{\mu\rho,\sigma} - \frac{1}{2} h_{\rho\sigma,\mu} + (D-2) H a h_{\mu\rho} \delta^0_\sigma \Big) . \quad (60)$$

Because our gauge condition breaks de Sitter invariance it will be necessary to contemplate noninvariant counterterms. It is therefore appropriate to digress at this point with a description of the various de Sitter symmetries and their effect upon Equation 60. In our D-dimensional conformal coordinate system the $\frac{1}{2}D(D{+}1)$ de Sitter transformations take the following form:

1. Spatial translations — comprising $(D-1)$ transformations.

$$\eta' = \eta \, , \quad (61)$$
$$x'^i = x^i + \epsilon^i \, . \quad (62)$$

2. Rotations — comprising $\frac{1}{2}(D-1)(D-2)$ transformations.

$$\eta' = \eta \, , \quad (63)$$
$$x'^i = R^{ij} x^j \, . \quad (64)$$

3. Dilatation — comprising 1 transformation.

$$\eta' = k \, \eta \, , \quad (65)$$
$$x'^i = k \, x^i \, . \quad (66)$$

4. Spatial special conformal transformations — comprising $(D-1)$ transformations.

$$\eta' = \frac{\eta}{1-2\vec{\theta}\cdot\vec{x}+\|\vec{\theta}\|^2 x\cdot x} , \qquad (67)$$

$$x'^i = \frac{x^i - \theta^i x\cdot x}{1-2\vec{\theta}\cdot\vec{x}+\|\vec{\theta}\|^2 x\cdot x} . \qquad (68)$$

It is easy to check that our gauge condition respects all of these but the spatial special conformal transformations. We will see that the other symmetries impose important restrictions upon the BPHZ counterterms which are allowed.

It is now time to solve for the graviton propagator. Because its space and time components are treated differently in our coordinate system and gauge it is useful to have an expression for the purely spatial parts of the Lorentz metric and the Kronecker delta,

$$\overline{\eta}_{\mu\nu} \equiv \eta_{\mu\nu} + \delta^0_\mu \delta^0_\nu \qquad \text{and} \qquad \overline{\delta}^\mu_\nu \equiv \delta^\mu_\nu - \delta^\mu_0 \delta^0_\nu . \qquad (69)$$

The quadratic part of $\mathcal{L}_{\rm Einstein} + \mathcal{L}_{GF}$ can be partially integrated to take the form $\frac{1}{2} h^{\mu\nu} D_{\mu\nu}{}^{\rho\sigma} h_{\rho\sigma}$, where the kinetic operator is,

$$D_{\mu\nu}{}^{\rho\sigma} \equiv \left\{ \frac{1}{2} \overline{\delta}_\mu{}^{(\rho} \overline{\delta}_\nu{}^{\sigma)} - \frac{1}{4} \eta_{\mu\nu} \eta^{\rho\sigma} - \frac{1}{2(D-3)} \delta^0_\mu \delta^0_\nu \delta^\rho_0 \delta^\sigma_0 \right\} D_A$$
$$+ \delta^0_{(\mu} \overline{\delta}^{(\rho}_{\nu)} \delta^{\sigma)}_0 \, D_B + \frac{1}{2} \Big(\frac{D-2}{D-3}\Big) \delta^0_\mu \delta^0_\nu \delta^\rho_0 \delta^\sigma_0 \, D_C , \qquad (70)$$

and the three scalar differential operators are,

$$D_A \equiv \partial_\mu \Big(\sqrt{-g} g^{\mu\nu} \partial_\nu \Big) , \qquad (71)$$

$$D_B \equiv \partial_\mu \Big(\sqrt{-g} g^{\mu\nu} \partial_\nu \Big) - \frac{1}{D} \Big(\frac{D-2}{D-1} \Big) R \sqrt{-g} , \qquad (72)$$

$$D_C \equiv \partial_\mu \Big(\sqrt{-g} g^{\mu\nu} \partial_\nu \Big) - \frac{2}{D} \Big(\frac{D-3}{D-1} \Big) R \sqrt{-g} . \qquad (73)$$

The graviton propagator in this gauge takes the form of a sum of constant index factors times scalar propagators,

$$i\Big[{}_{\mu\nu}\Delta_{\rho\sigma}\Big](x;x') = \sum_{I=A,B,C} \Big[{}_{\mu\nu} T^I_{\rho\sigma}\Big] i\Delta_I(x;x') . \qquad (74)$$

The three scalar propagators invert the various scalar kinetic operators,

$$D_I \times i\Delta_I(x;x') = i\delta^D(x-x') \qquad \text{for} \qquad I = A, B, C \; , \tag{75}$$

and we will presently give explicit expressions for them. The index factors are,

$$\begin{aligned}
\Big[{}_{\mu\nu}T^A_{\rho\sigma}\Big] &= 2\,\overline{\eta}_{\mu(\rho}\overline{\eta}_{\sigma)\nu} - \frac{2}{D\!-\!3}\overline{\eta}_{\mu\nu}\overline{\eta}_{\rho\sigma} \; , &(76)\\
\Big[{}_{\mu\nu}T^B_{\rho\sigma}\Big] &= -4\delta^0_{(\mu}\overline{\eta}_{\nu)(\rho}\delta^0_{\sigma)} \; , &(77)\\
\Big[{}_{\mu\nu}T^C_{\rho\sigma}\Big] &= \frac{2}{(D\!-\!2)(D\!-\!3)}\Big[(D\!-\!3)\delta^0_\mu\delta^0_\nu + \overline{\eta}_{\mu\nu}\Big]\Big[(D\!-\!3)\delta^0_\rho\delta^0_\sigma + \overline{\eta}_{\rho\sigma}\Big] \; . &(78)
\end{aligned}$$

With these definitions and Equation 75 for the scalar propagators it is straightforward to verify that the graviton propagator of Equation 74 indeed inverts the gauge-fixed kinetic operator,

$$D_{\mu\nu}{}^{\rho\sigma} \times i\Big[{}_{\rho\sigma}\Delta^{\alpha\beta}\Big](x;x') = \delta^{(\alpha}_\mu \delta^{\beta)}_\nu i\delta^D(x-x') \; . \tag{79}$$

The scalar propagators can be expressed in terms of the following function of the invariant length $\ell(x;x')$ between x^μ and x'^μ,

$$\begin{aligned}
y(x;x') &\equiv 4\sin^2\Big(\frac{1}{2}H\ell(x;x')\Big) = aa'H^2\Delta x^2(x;x') \; , &(80)\\
&= aa'H^2\Big(\|\vec{x}-\vec{x}'\|^2 - (|\eta\!-\!\eta'|\!-\!i\delta)^2\Big) \; . &(81)
\end{aligned}$$

The most singular term for each case is the propagator for a massless, conformally coupled scalar [53],

$$i\Delta_{\rm cf}(x;x') = \frac{H^{D-2}}{(4\pi)^{\frac{D}{2}}}\Gamma\Big(\frac{D}{2}\!-\!1\Big)\Big(\frac{4}{y}\Big)^{\frac{D}{2}-1} \; . \tag{82}$$

The A-type propagator obeys the same equation as that of a massless, minimally coupled scalar. It has long been known that no de Sitter invariant solution exists [54]. If one elects to break de Sitter invariance while preserving homogeneity of Equations 61-62 and isotropy of Equations 63-64 — this is known as the "E(3)" vacuum [55] — the minimal solution is [56, 57],

$$\begin{aligned}
i\Delta_A(x;x') &= i\Delta_{\rm cf}(x;x')\\
&\quad + \frac{H^{D-2}}{(4\pi)^{\frac{D}{2}}}\frac{\Gamma(D\!-\!1)}{\Gamma(\frac{D}{2})}\left\{\frac{D}{D\!-\!4}\frac{\Gamma^2(\frac{D}{2})}{\Gamma(D\!-\!1)}\Big(\frac{4}{y}\Big)^{\frac{D}{2}-2} - \pi\cot\Big(\frac{\pi}{2}D\Big) + \ln(aa')\right\}\\
&\quad + \frac{H^{D-2}}{(4\pi)^{\frac{D}{2}}}\sum_{n=1}^{\infty}\left\{\frac{1}{n}\frac{\Gamma(n\!+\!D\!-\!1)}{\Gamma(n\!+\!\frac{D}{2})}\Big(\frac{y}{4}\Big)^n - \frac{1}{n\!-\!\frac{D}{2}\!+\!2}\frac{\Gamma(n\!+\!\frac{D}{2}\!+\!1)}{\Gamma(n\!+\!2)}\Big(\frac{y}{4}\Big)^{n-\frac{D}{2}+2}\right\} .
\end{aligned} \tag{83}$$

Note that this solution breaks dilatation invariance of Equations 65-66 in addition to the spatial special conformal invariance of Equations 67-68 broken by the gauge condition. By convoluting naive de Sitter transformations with the compensating diffeomorphisms necessary to restore our gauge condition of Equation 60 one can show that the breaking of dilatation invariance is physical whereas the apparent breaking of spatial special conformal invariance is a gauge artifact [58].

The B-type and C-type propagators possess de Sitter invariant (and also unique) solutions,

$$i\Delta_B(x;x') = i\Delta_{\rm cf}(x;x') - \frac{H^{D-2}}{(4\pi)^{\frac{D}{2}}} \sum_{n=0}^{\infty} \Bigg\{ \frac{\Gamma(n+D-2)}{\Gamma(n+\frac{D}{2})} \Big(\frac{y}{4}\Big)^n - \frac{\Gamma(n+\frac{D}{2})}{\Gamma(n+2)} \Big(\frac{y}{4}\Big)^{n-\frac{D}{2}+2} \Bigg\}, \qquad (84)$$

$$i\Delta_C(x;x') = i\Delta_{\rm cf}(x;x') + \frac{H^{D-2}}{(4\pi)^{\frac{D}{2}}} \sum_{n=0}^{\infty} \Bigg\{ (n+1) \frac{\Gamma(n+D-3)}{\Gamma(n+\frac{D}{2})} \Big(\frac{y}{4}\Big)^n - \Big(n-\frac{D}{2}+3\Big) \frac{\Gamma(n+\frac{D}{2}-1)}{\Gamma(n+2)} \Big(\frac{y}{4}\Big)^{n-\frac{D}{2}+2} \Bigg\}. \qquad (85)$$

They can be more compactly, but less usefully, expressed as hypergeometric functions [59, 60],

$$i\Delta_B(x;x') = \frac{H^{D-2}}{(4\pi)^{\frac{D}{2}}} \frac{\Gamma(D-2)\Gamma(1)}{\Gamma(\frac{D}{2})} \, {}_2F_1\Big(D-2, 1; \frac{D}{2}; 1-\frac{y}{4}\Big) \; , \qquad (86)$$

$$i\Delta_C(x;x') = \frac{H^{D-2}}{(4\pi)^{\frac{D}{2}}} \frac{\Gamma(D-3)\Gamma(2)}{\Gamma(\frac{D}{2})} \, {}_2F_1\Big(D-3, 2; \frac{D}{2}; 1-\frac{y}{4}\Big) \; . \qquad (87)$$

These expressions might seem daunting but they are actually simple to use because the infinite sums vanish in $D = 4$, and each term in these sums goes like a positive power of $y(x;x')$. This means the infinite sums can only contribute when multiplied by a divergent term, and even then only a small number of terms can contribute. Note also that the B-type and C-type propagators agree with the conformal propagator in $D = 4$.

In view of the subtle problems associated with the graviton propagator in what seemed to be perfectly valid, de Sitter invariant gauges [46, 41], it is well to review the extensive checks that have been made on the consistency of this noninvariant propagator. On the classical level it has been checked

that the response to a point mass is in perfect agreement with the linearized, de Sitter-Schwarzchild geometry [41]. The linearized diffeomorphisms which enforce the gauge condition have also been explicitly constructed [61]. Although a tractable, D-dimensional form for the various scalar propagators $i\Delta_I(x;x')$ was not originally known, some simple identities obeyed by the mode functions in their Fourier expansions sufficed to verify the tree order Ward identity [61]. The full, D-dimensional formalism has been used recently to compute the graviton 1-point function at one loop order [62]. The result seems to be in qualitative agreement with canonical computations in other gauges [63, 64]. A $D=3+1$ version of the formalism — with regularization accomplished by keeping the parameter $\delta \neq 0$ in the de Sitter length function $y(x;x')$ Equation 81 — was used to evaluate the leading late time correction to the 2-loop 1-point function [65, 66]. The same technique was used to compute the unrenormalized graviton self-energy at one loop order [13]. An explicit check was made that the flat space limit of this quantity agrees with Capper's result [67] for the graviton self-energy in the same gauge. The one loop Ward identity was also checked in de Sitter background [13]. Finally, the $D=4$ formalism was used to compute the two loop contribution from a massless, minimally coupled scalar to the 1-graviton function [68]. The result was shown to obey an important bound imposed by global conformal invariance on the maximum possible late time effect.

2.3 Renormalization and Counterterms

It remains to deal with the local counterterms we must add, order-by-order in perturbation theory, to absorb divergences in the sense of BPHZ renormalization. The particular counterterms which renormalize the fermion self-energy must obviously involve a single $\overline{\psi}$ and a single ψ.[3] At one loop order the superficial degree of divergence of quantum gravitational contributions to the fermion self-energy is three, so the necessary counterterms can involve zero, one, two or three derivatives. These derivatives can either act upon the fermi fields or upon the metric, in which case they must be organized into curvatures or derivatives of curvatures. We will first exhaust the possible invariant counterterms for a general renormalized fermion mass and a

[3] Although the Dirac Lagrangian is conformally invariant, the counterterms required to renormalize the fermion self-energy will not possess this symmetry because quantum gravity does not. We must therefore work with the original fields rather than the conformally rescaled ones.

general background geometry, and then specialize to the case of zero mass in de Sitter background. We close with a discussion of possible noninvariant counterterms.

All one loop corrections from quantum gravity must carry a factor of $\kappa^2 \sim$ mass^{-2}. There will be additional dimensions associated with derivatives and with the various fields, and the balance must be struck using the renormalized fermion mass, m. Hence the only invariant counterterm with no derivatives has the form,

$$\kappa^2 m^3 \overline{\psi} \psi \sqrt{-g} \; . \tag{88}$$

With one derivative we can always partially integrate to act upon the ψ field, so the only invariant counterterm is,

$$\kappa^2 m^2 \overline{\psi} i \not{\mathcal{D}} \psi \sqrt{-g} \; . \tag{89}$$

Two derivatives can either act upon the fermions or else on the metric to produce curvatures. We can organize the various possibilities as follows,

$$\kappa^2 m \overline{\psi} (i \not{\mathcal{D}})^2 \psi \sqrt{-g} \quad , \quad \kappa^2 m R \overline{\psi} \psi \sqrt{-g} \; . \tag{90}$$

Three derivatives can be all acted on the fermions, or one on the fermions and two in the form of curvatures, or there can be a differentiated curvature,

$$\kappa^2 \overline{\psi} \Bigl((i \not{\mathcal{D}})^2 + \frac{R}{D(D-1)} \Bigr) i \not{\mathcal{D}} \psi \sqrt{-g} \quad , \quad \kappa^2 R \, \overline{\psi} \, i \not{\mathcal{D}} \psi \sqrt{-g} \; ,$$
$$\kappa^2 e_{\mu m} \Bigl(R^{\mu\nu} - \frac{1}{D} g^{\mu\nu} R \Bigr) \overline{\psi} \gamma^m i \mathcal{D}_\nu \psi \sqrt{-g} \quad , \quad \kappa^2 e^{\mu}{}_{m} R_{,\mu} \overline{\psi} \gamma^m \psi \sqrt{-g} \; . \tag{91}$$

Because mass is multiplicatively renormalized in dimensional regularization, and because we are dealing with zero mass fermions, counterterms in Equations 88, 89 and 90 are all unnecessary for our calculation. Although all four counterterms of Equation 91 are nonzero and distinct for a general metric background, they only affect our fermion self-energy for the special case of de Sitter background. For that case $R_{\mu\nu} = (D-1) H^2 g_{\mu\nu}$, so the last two counterterms vanish. The specialization of the invariant counter-Lagrangian we require to de Sitter background is therefore,

$$\Delta \mathcal{L}_{\rm inv} = \alpha_1 \kappa^2 \overline{\psi} \Bigl((i \not{\mathcal{D}})^2 + \frac{R}{D(D-1)} \Bigr) i \not{\mathcal{D}} \psi \sqrt{-g} + \alpha_2 \kappa^2 R \, \overline{\psi} \, i \not{\mathcal{D}} \psi \sqrt{-g} \; , \tag{92}$$

$$\longrightarrow \alpha_1 \kappa^2 \overline{\Psi} \Bigl(i \not{\partial} a^{-1} i \not{\partial} a^{-1} + \frac{R}{D(D-1)} \Bigr) i \not{\partial} \Psi + \alpha_2 (D-1) D \kappa^2 H^2 \overline{\Psi} i \not{\partial} \Psi \; . \tag{93}$$

Here α_1 and α_2 are D-dependent constants which are dimensionless for $D=4$. The associated vertex operators are,

$$C_{1ij} \equiv \alpha_1\kappa^2\Big(i\partial\!\!\!/ a^{-1}i\partial\!\!\!/ a^{-1}i\partial\!\!\!/ + H^2 i\partial\!\!\!/\Big)_{ij} = \alpha_1\kappa^2\Big(a^{-1}i\partial\!\!\!/ \partial^2 a^{-1}\Big)_{ij} , \quad (94)$$

$$C_{2ij} \equiv \alpha_2(D-1)D\kappa^2 H^2 i\partial\!\!\!/_{ij} . \quad (95)$$

Of course C_1 is the higher derivative counterterm mentioned in section 1. It will renormalize the most singular terms — coming from the $i\Delta_{\rm cf}$ part of the graviton propagator — which are unimportant because they are suppressed by powers of the scale factor. The other vertex operator, C_2, is a sort of dimensionful field strength renormalization in de Sitter background. It will renormalize the less singular contributions which derive physically from inflationary particle production.

The one loop fermion self-energy would require no additional counterterms had it been possible to use the background field technique in background field gauge [69, 70, 71, 72]. However, the obstacle to using a de Sitter invariant gauge obviously precludes this. We must therefore come to terms with the possibility that divergences may arise which require noninvariant counterterms. What form can these counterterms take? Applying the BPHZ theorem [19, 20, 21, 22] to the gauge-fixed theory in de Sitter background implies that the relevant counterterms must still consist of κ^2 times a spinor differential operator with the dimension of mass-cubed, involving no more than three derivatives and acting between $\overline{\Psi}$ and Ψ. As the only dimensionful constant in our problem, powers of H must be used to make up whatever dimensions are not supplied by derivatives.

Because dimensional regularization respects diffeomorphism invariance, it is only the gauge fixing term in Equation 60 that permits noninvariant counterterms.[4] Conversely, noninvariant counterterms must respect the residual symmetries of the gauge condition. Homogeneity of Equations 61-62 implies

[4]One might think that the they could come as well from the fact that the vacuum breaks de Sitter invariance, but symmetries broken by the vacuum do not introduce new counterterms [73]. Highly relevant, explicit examples are provided by recent computations for a massless, minimally coupled scalar with a quartic self-interaction in the same locally de Sitter background used here. The vacuum in this theory also breaks de Sitter invariance but noninvariant counterterms fail to arise even at *two loop* order in either the expectation value of the stress tensor [56, 57] or the self-mass-squared [17]. It is also relevant that the one loop vacuum polarization from (massless, minimally coupled) scalar quantum electrodynamics is free of noninvariant counterterms in the same background [9].

that the spinor differential operator cannot depend upon the spatial coordinate x^i. Similarly, isotropy of Equations 63-64 requires that any spatial derivative operators ∂_i must either be contracted into γ^i or another spatial derivative. Owing to the identity,

$$(\gamma^i \partial_i)^2 = -\nabla^2 \; , \tag{96}$$

we can think of all spatial derivatives as contracted into γ^i. Although the temporal derivative is not required to be multiplied by γ^0 we lose nothing by doing so provided additional dependence upon γ^0 is allowed.

The final residual symmetry is dilatation invariance shown by Equations 65-66. It has the crucial consequence that derivative operators can only appear in the form $a^{-1}\partial_\mu$. In addition the entire counterterm must have an overall factor of a, and there can be no other dependence upon η. So the most general counterterm consistent with our gauge condition takes the form,

$$\Delta \mathcal{L}_{\rm non} = \kappa^2 H^3 a \overline{\Psi} \mathcal{S}\Bigl((Ha)^{-1} \gamma^0 \partial_0 , (Ha)^{-1} \gamma^i \partial_i \Bigr) \Psi \; , \tag{97}$$

where the spinor function $\mathcal{S}(b,c)$ is at most a third order polynomial function of its arguments, and it may involve γ^0 in an arbitrary way.

Three more principles constrain noninvariant counterterms. The first of these principles is that the fermion self-energy involves only odd powers of gamma matrices. This follows from the masslessness of our fermion and the consequent fact that the fermion propagator and each interaction vertex involves only odd numbers of gamma matrices. This principle fixes the dependence upon γ^0 and allows us to express the spinor differential operator in terms of just ten constants β_i,

$$\begin{aligned}
\kappa^2 H^3 a \mathcal{S}\Bigl((Ha)^{-1} \gamma^0 \partial_0 , (Ha)^{-1} \gamma^i \partial_i \Bigr) = \kappa^2 a \Biggl\{ \beta_1 (a^{-1} \gamma^0 \partial_0)^3 \\
+ \beta_2 \Bigl[(a^{-1} \gamma^0 \partial_0)^2 (a^{-1} \gamma^i \partial_i) \Bigr] + \beta_3 \Bigl[(a^{-1} \gamma^0 \partial_0) (a^{-1} \gamma^i \partial_i)^2 \Bigr] + \beta_4 (a^{-1} \gamma^i \partial_i)^3 \\
+ H \gamma^0 \Biggl(\beta_5 (a^{-1} \gamma^0 \partial_0)^2 + \beta_6 \Bigl[(a^{-1} \gamma^0 \partial_0) (a^{-1} \gamma^i \partial_i) \Bigr] + \beta_7 (a^{-1} \gamma^i \partial_i)^2 \Biggr) \\
+ H^2 \Biggl(\beta_8 (a^{-1} \gamma^0 \partial_0) + \beta_9 (a^{-1} \gamma^i \partial_i) \Biggr) + H^3 \gamma^0 \beta_{10} \Biggr\} .
\end{aligned} \tag{98}$$

In this expansion, but for the rest of this section only, we define noncommuting factors within square brackets to be symmetrically ordered, for example,

$$\Bigl[(a^{-1} \gamma^0 \partial_0)^2 (a^{-1} \gamma^i \partial_i) \Bigr] \equiv \frac{1}{3} (a^{-1} \gamma^0 \partial_0)^2 (a^{-1} \gamma^i \partial_i)$$

$$+\frac{1}{3}(a^{-1}\gamma^0\partial_0)(a^{-1}\gamma^i\partial_i)(a^{-1}\gamma^0\partial_0)+\frac{1}{3}(a^{-1}\gamma^i\partial_i)(a^{-1}\gamma^0\partial_0)^2 \; . \quad (99)$$

The second principle is that our gauge condition of Equation 60 becomes Poincaré invariant in the flat space limit of $H \rightarrow 0$, where the conformal time is $\eta = -e^{-Ht}/H$ with t held fixed. In that limit only the four cubic terms of Equation 98 survive,

$$\lim_{H\rightarrow 0} \kappa^2 H^3 a \mathcal{S}\Bigl((Ha)^{-1}\gamma^0\partial_0, (Ha)^{-1}\gamma^i\partial_i\Bigr) = \kappa^2 \Biggl\{\beta_1(\gamma^0\partial_0)^3 \\ +\beta_2\Bigl[(\gamma^0\partial_0)^2(\gamma^i\partial_i)\Bigr]+\beta_3\Bigl[(\gamma^0\partial_0)(\gamma^i\partial_i)^2\Bigr]+\beta_4(\gamma^i\partial_i)^3\Biggr\}. \quad (100)$$

Because the entire theory is Poincaré invariant in that limit, these four terms must sum to a term proportional to $(\gamma^\mu\partial_\mu)^3$, which implies,

$$\beta_1 = \frac{1}{3}\beta_2 = \frac{1}{3}\beta_3 = \beta_4 \; . \quad (101)$$

But in that case the four cubic terms sum to give a linear combination of the invariant counterterms of Equation 94 and Equation 95,

$$\kappa^2 a \Biggl\{(a^{-1}\gamma^0\partial_0)^3 + 3\Bigl[(a^{-1}\gamma^0\partial_0)^2(a^{-1}\gamma^i\partial_i)\Bigr] \\ +3\Bigl[(a^{-1}\gamma^0\partial_0)(a^{-1}\gamma^i\partial_i)^2\Bigr]+(a^{-1}\gamma^i\partial_i)^3\Biggr\} = \kappa^2 \,\partial\!\!\!/\, a^{-1} \partial\!\!\!/ a^{-1} \partial\!\!\!/ \; . \quad (102)$$

Because we have already counted this combination among the invariant counterterms it need not be included in $\mathcal{S}$.

The final simplifying principle is that the fermion self-energy is odd under interchange of x^μ and x'^μ,

$$-i\Bigl[{}_i\Sigma_j\Bigr](x;x') = +i\Bigl[{}_i\Sigma_j\Bigr](x';x) \; . \quad (103)$$

This symmetry is trivial at tree order, but not easy to show generally. Moreover, it isn't a property of individual terms, many of which violate Equation 103. However, when everything is summed up the result must obey Equation 103, hence so too must the counterterms. This has the immediate consequence of eliminating the counterterms with an even number of derivatives: those proportional to β_{5-7} and to β_{10}. We have already dispensed with β_{1-4},

which leaves only the linear terms, β_{8-9}. Because one linear combination of these already appears in the invariant of Equation 95 the sole noninvariant counterterm we require is,

$$\Delta\mathcal{L}_{\rm non} = \overline{\Psi} C_3 \Psi \qquad \text{where} \qquad C_{3ij} \equiv \alpha_3 \kappa^2 H^2 i \overline{\partial\!\!\!/}_{ij} \; . \tag{104}$$

3 COMPUTATIONAL RESULTS

For one-loop order the big simplification of working in position space is that it doesn't involve any integrations after all the delta functions are used. However, even though calculating the one loop fermion self energy is only a multiplication of propagators, vertices and derivatives, the computation is still a tedious work owing to the great number of vertices and the complicated graviton propagator. Generally speaking, we first contract 4-point and pairs of 3-point vertices into the full graviton propagator. Then we break up the graviton propagator into its conformal part plus the residuals proportional to each of three index factors. The next step is to act the derivatives and sum up the results. At each step we also tabulate the results in order to clearly see the potential tendencies such as cancelations among these terms. Finally, we must remember that the fermion self energy will be used inside an integral in the quantum-corrected Dirac equation. For this purpose, we extract the derivatives with respect to the coordinates "x^μ" by partially integrating them out. This procedure also can be implemented so as to segregate the divergence to a delta function that can be absorbed by the counterterms which we found in chapter 2.

3.1 Contributions from the 4-Point Vertices

In this section we evaluate the contributions from 4-point vertex operators of Table 1. The generic diagram topology is depicted in Figure 1. The analytic form is,

$$-i\Big[{}_i\Sigma^{\rm 4pt}_j\Big](x;x') = \sum_{I=1}^{8} iU^{\alpha\beta\rho\sigma}_{Iij} \, i\Big[{}_{\alpha\beta}\Delta_{\rho\sigma}\Big](x;x')\, \delta^D(x-x') \; . \tag{105}$$

And the generic contraction for each of the vertex operators in Table 1 is given in Table 2.

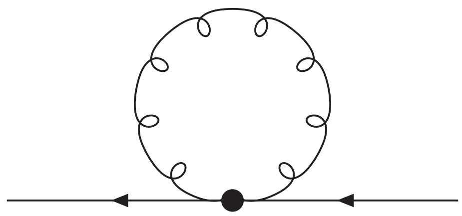

Figure 1: Contribution from 4-point vertices.

Table 2: Generic 4-point contractions

I	$i[{}_{\alpha\beta}\Delta_{\rho\sigma}](x;x')\, iU_I^{\alpha\beta\rho\sigma}\, \delta^D(x-x')$
1	$-\frac{1}{8}\kappa^2\, i[{}^{\alpha}{}_{\alpha}\Delta^{\rho}{}_{\rho}](x;x)\, \not{\partial}\, \delta^D(x-x')$
2	$\frac{1}{4}\kappa^2\, i[{}^{\alpha\beta}\Delta_{\alpha\beta}](x;x)\, \not{\partial}\, \delta^D(x-x')$
3	$\frac{1}{4}\kappa^2\, i[{}^{\alpha}{}_{\alpha}\Delta_{\rho\sigma}](x;x)\, \gamma^\rho\partial^\sigma\, \delta^D(x-x')$
4	$-\frac{3}{8}\kappa^2\, i[{}^{\alpha}{}_{\beta}\Delta_{\alpha\sigma}](x;x)\, \gamma^\beta\partial^\sigma\, \delta^D(x-x')$
5	$-\frac{i}{4}\kappa^2\, \partial'_\mu i[{}^{\alpha}{}_{\alpha}\Delta_{\rho\sigma}](x;x')\, \gamma^\rho J^{\sigma\mu}\, \delta^D(x-x')$
6	$\frac{i}{8}\kappa^2\, \partial'_\mu i[{}^{\alpha}{}_{\beta}\Delta_{\alpha\sigma}](x;x')\, \gamma^\mu J^{\beta\sigma}\, \delta^D(x-x')$
7	$\frac{i}{4}\kappa^2\, \partial_\mu i[{}^{\alpha}{}_{\beta}\Delta_{\alpha\sigma}](x;x)\, \gamma^\beta J^{\sigma\mu}\, \delta^D(x-x')$
8	$\frac{i}{4}\kappa^2\, \partial'^\beta i[{}_{\alpha\beta}\Delta_{\rho\sigma}](x;x')\, \gamma^\rho J^{\sigma\alpha}\, \delta^D(x-x')$

From an examination of the generic contractions in Table 2 it is apparent that we must work out how the three index factors $[{}_{\alpha\beta}T^I_{\rho\sigma}]$ which make up the graviton propagator contract into $\eta^{\alpha\beta}$ and $\eta^{\alpha\rho}$. For the A-type and B-type index factors the various contractions give,

$$\eta^{\alpha\beta}\Big[{}_{\alpha\beta}T^A_{\rho\sigma}\Big] = -\Big(\frac{4}{D-3}\Big)\overline{\eta}_{\rho\sigma} \quad , \quad \eta^{\alpha\rho}\Big[{}_{\alpha\beta}T^A_{\rho\sigma}\Big] = \Big(D - \frac{2}{D-3}\Big)\overline{\eta}_{\beta\sigma}\ , \tag{106}$$

$$\eta^{\alpha\beta}\Big[{}_{\alpha\beta}T^B_{\rho\sigma}\Big] = 0 \quad , \quad \eta^{\alpha\rho}\Big[{}_{\alpha\beta}T^B_{\rho\sigma}\Big] = -(D-1)\,\delta^0_\beta\delta^0_\sigma + \overline{\eta}_{\beta\sigma}, \tag{107}$$

For the C-type index factor they are,

$$\eta^{\alpha\beta}\left[{}_{\alpha\beta}T^C_{\rho\sigma}\right] = \Bigl(\frac{4}{D-2}\Bigr)\,\delta^0_\rho\delta^0_\sigma + \frac{4}{(D-2)(D-3)}\,\overline{\eta}_{\rho\sigma}\ ,$$

$$\eta^{\alpha\rho}\left[{}_{\alpha\beta}T^C_{\rho\sigma}\right] = -2\Bigl(\frac{D-3}{D-2}\Bigr)\,\delta^0_\beta\delta^0_\sigma + \frac{2}{(D-2)(D-3)}\,\overline{\eta}_{\beta\sigma}\ . \qquad (108)$$

On occasion we also require double contractions. For the A-type index factor these are,

$$\eta^{\alpha\beta}\eta^{\rho\sigma}\left[{}_{\alpha\beta}T^A_{\rho\sigma}\right] = -4\Bigl(\frac{D-1}{D-3}\Bigr)\ ,$$

$$\eta^{\alpha\rho}\eta^{\beta\sigma}\left[{}_{\alpha\beta}T^A_{\rho\sigma}\right] = D(D-1) - 2\Bigl(\frac{D-1}{D-3}\Bigr)\ . \qquad (109)$$

The double contractions of the B-type and C-type index factors are,

$$\eta^{\alpha\beta}\eta^{\rho\sigma}\left[{}_{\alpha\beta}T^B_{\rho\sigma}\right] = 0 \quad , \quad \eta^{\alpha\rho}\eta^{\beta\sigma}\left[{}_{\alpha\beta}T^B_{\rho\sigma}\right] = 2(D-1)\ , \qquad (110)$$

$$\eta^{\alpha\beta}\eta^{\rho\sigma}\left[{}_{\alpha\beta}T^C_{\rho\sigma}\right] = \frac{8}{(D-2)(D-3)} \quad , \quad \eta^{\alpha\rho}\eta^{\beta\sigma}\left[{}_{\alpha\beta}T^C_{\rho\sigma}\right] = 2\frac{(D^2-5D+8)}{(D-2)(D-3)}. \qquad (111)$$

Table 3 was generated from Table 2 by expanding the graviton propagator in terms of index factors,

$$i\Bigl[{}_{\alpha\beta}\Delta_{\rho\sigma}\Bigr](x;x') = \Bigl[{}_{\alpha\beta}T^A_{\rho\sigma}\Bigr] i\Delta_A(x;x') + \Bigl[{}_{\alpha\beta}T^B_{\rho\sigma}\Bigr] i\Delta_B(x;x') + \Bigl[{}_{\alpha\beta}T^C_{\rho\sigma}\Bigr] i\Delta_C(x;x')\ . \qquad (112)$$

We then perform the relevant contractions using the previous identities. Relation 32 was also exploited to simplify the gamma matrix structure.

From Table 3 it is apparent that we require the coincidence limits of zero or one derivatives acting on each of the scalar propagators. For the A-type propagator these are,

$$\lim_{x'\to x} i\Delta_A(x;x') = \frac{H^{D-2}}{(4\pi)^{\frac{D}{2}}}\frac{\Gamma(D-1)}{\Gamma(\frac{D}{2})}\left\{-\pi\cot\Bigl(\frac{\pi}{2}D\Bigr) + 2\ln(a)\right\}, \qquad (113)$$

$$\lim_{x'\to x} \partial_\mu i\Delta_A(x;x') = \frac{H^{D-2}}{(4\pi)^{\frac{D}{2}}}\frac{\Gamma(D-1)}{\Gamma(\frac{D}{2})} \times Ha\delta^0_\mu\ . \qquad (114)$$

The analogous coincidence limits for the B-type propagator are actually finite in $D=4$ dimensions,

$$\lim_{x'\to x} i\Delta_B(x;x') = \frac{H^{D-2}}{(4\pi)^{\frac{D}{2}}}\frac{\Gamma(D-1)}{\Gamma(\frac{D}{2})} \times -\frac{1}{D-2}\ , \qquad (115)$$

$$\lim_{x'\to x} \partial_\mu i\Delta_B(x;x') = 0\ . \qquad (116)$$

Table 3: Four-point contribution from each part of the graviton propagator.

I	J	$i[{}_{\alpha\beta}T^{J}_{\rho\sigma}]\, i\Delta_J(x;x')\, iU_I^{\alpha\beta\rho\sigma}\, \delta^D(x-x')$
1	A	$\frac{1}{2}(\frac{D-1}{D-3})\kappa^2\, i\Delta_A(x;x)\, \partial\!\!\!/\, \delta^D(x-x')$
1	B	0
1	C	$-\frac{1}{(D-2)(D-3)}\kappa^2\, i\Delta_C(x;x)\, \partial\!\!\!/\, \delta^D(x-x')$
2	A	$(\frac{D-1}{4})(\frac{D^2-3D-2}{D-3})\kappa^2\, i\Delta_A(x;x)\, \partial\!\!\!/\, \delta^D(x-x')$
2	B	$(\frac{D-1}{2})\kappa^2\, i\Delta_B(x;x)\, \partial\!\!\!/\, \delta^D(x-x')$
2	C	$\frac{1}{2}\frac{(D^2-5D+8)}{(D-2)(D-3)}\kappa^2\, i\Delta_C(x;x)\, \partial\!\!\!/\, \delta^D(x-x')$
3	A	$-\frac{1}{D-3}\kappa^2\, i\Delta_A(x;x)\overline{\partial\!\!\!/}\, \delta^D(x-x')$
3	B	0
3	C	$\frac{1}{(D-2)(D-3)}\kappa^2\, i\Delta_C(x;x)[\overline{\partial\!\!\!/}-(D-3)\gamma^0\partial_0]\delta^D(x-x')$
4	A	$-\frac{3}{8}(\frac{D^2-3D-2}{D-3})\kappa^2\, i\Delta_A(x;x)\overline{\partial\!\!\!/}\, \delta^D(x-x')$
4	B	$-\frac{3}{8}\kappa^2\, i\Delta_B(x;x)[\overline{\partial\!\!\!/}+(D-1)\gamma^0\partial_0]\delta^D(x-x')$
4	C	$-\frac{3}{4}\frac{1}{(D-2)(D-3)}\kappa^2\, i\Delta_C(x;x)[\overline{\partial\!\!\!/}+(D-3)^2\gamma^0\partial_0]\delta^D(x-x')$
5	A	$\kappa^2[-\frac{1}{2(D-3)}\overline{\partial\!\!\!/}'+\frac{1}{2}(\frac{D-1}{D-3})\partial\!\!\!/']\, i\Delta_A(x;x')\, \delta^D(x-x')$
5	B	0
5	C	$-\frac{1}{(D-2)(D-3)}\kappa^2\, [\frac{1}{2}\overline{\partial\!\!\!/}'+(\frac{D-1}{2})\gamma^0\partial_0']\, i\Delta_C(x;x')\, \delta^D(x-x')$
6	A	0
6	B	0
6	C	0
7	A	$(\frac{D^2-3D-2}{D-3})\kappa^2[-\frac{1}{8}\overline{\partial\!\!\!/}+(\frac{D-1}{8})\partial\!\!\!/]\, i\Delta_A(x;x)\, \delta^D(x-x')$
7	B	$\kappa^2[(\frac{D-2}{8})\overline{\partial\!\!\!/}+(\frac{D-1}{8})\partial\!\!\!/]\, i\Delta_B(x;x)\, \delta^D(x-x')$
7	C	$\frac{1}{4}\kappa^2[\frac{(D^2-6D+8)}{(D-2)(D-3)}\overline{\partial\!\!\!/}+\frac{(D-1)}{(D-2)(D-3)}\partial\!\!\!/]\, i\Delta_C(x;x)\, \delta^D(x-x')$
8	A	$-\kappa^2\frac{(D-2)(D-1)}{8(D-3)}\overline{\partial\!\!\!/}'\, i\Delta_A(x;x')\, \delta^D(x-x')$
8	B	$-\kappa^2[\frac{1}{8}\overline{\partial\!\!\!/}'+(\frac{D-1}{8})\gamma^0\partial_0']\, i\Delta_B(x;x')\, \delta^D(x-x')$
8	C	$\frac{1}{4}\kappa^2[\frac{1}{(D-2)(D-3)}\overline{\partial\!\!\!/}'-(\frac{D-1}{D-2})\gamma^0\partial_0']\, i\Delta_C(x;x')\, \delta^D(x-x')$

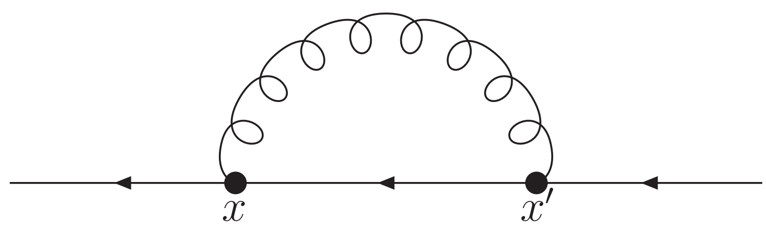

Figure 2: Contribution from two 3-point vertices.

The same is true for the coincidence limits of the C-type propagator,

$$\lim_{x'\to x} i\Delta_C(x;x') = \frac{H^{D-2}}{(4\pi)^{\frac{D}{2}}} \frac{\Gamma(D-1)}{\Gamma(\frac{D}{2})} \times \frac{1}{(D-2)(D-3)} , \qquad (117)$$

$$\lim_{x'\to x} \partial_\mu i\Delta_C(x;x') = 0 . \qquad (118)$$

Our final result for the 4-point contributions is given in Table 4. It was obtained from Table 3 by using the previous coincidence limits. We have also always chosen to re-express conformal time derivatives thusly,

$$\gamma^0\partial_0 = \not{\partial} - \overline{\not{\partial}} . \qquad (119)$$

A final point concerns the fact that the terms in the final column of Table 4 do not obey the reflection symmetry. In the next section we will find the terms which exactly cancel these.

3.2 Contributions from the 3-Point Vertices

In this section we evaluate the contributions from two 3-point vertex operators. The generic diagram topology is depicted in Figure 2. The analytic form is,

$$-i\Bigl[{}_i\Sigma^{\rm 3pt}_j\Bigr](x;x') = \sum_{I=1}^{3} iV^{\alpha\beta}_{Iik}(x)\, i\Bigl[{}_kS_\ell\Bigr](x;x') \sum_{J=1}^{3} iV^{\rho\sigma}_{J\ell j}(x')\, i\Bigl[{}_{\alpha\beta}\Delta_{\rho\sigma}\Bigr](x;x') . \qquad (120)$$

Table 4: Final 4-point contributions. All contributions are multiplied by $\frac{\kappa^2 H^{D-2}}{(4\pi)^{\frac{D}{2}}} \frac{\Gamma(D-1)}{\Gamma(\frac{D}{2})}$. We define $A \equiv \frac{\pi}{2}\cot(\frac{\pi D}{2}) - \ln(a)$.

I	J	$\not{\partial}\,\delta^D(x-x')$	$\overline{\not{\partial}}\,\delta^D(x-x')$	$aH\gamma^0\,\delta^D(x-x')$
1	A	$-(\frac{D-1}{D-3})A$	0	0
1	B	0	0	0
1	C	$-\frac{1}{(D-2)^2(D-3)^2}$	0	0
2	A	$[-\frac{D(D-1)}{2} + (\frac{D-1}{D-3})]A$	0	0
2	B	$-\frac{1}{2}(\frac{D-1}{D-2})$	0	0
2	C	$\frac{1}{2}\frac{(D^2-5D+8)}{(D-2)^2(D-3)^2}$	0	0
3	A	0	$\frac{2}{D-3}A$	0
3	B	0	0	0
3	C	$-\frac{1}{(D-2)^2(D-3)}$	$\frac{1}{(D-2)(D-3)^2}$	0
4	A	0	$[\frac{3D}{4} - \frac{3}{2(D-3)}]A$	0
4	B	$\frac{3}{8}(\frac{D-1}{D-2})$	$-\frac{3}{8}$	0
4	C	$-\frac{3}{4(D-2)^2}$	$\frac{3}{4}\frac{(D^2-6D+8)}{(D-2)^2(D-3)^2}$	0
5	A	0	0	$\frac{1}{2}(\frac{D-1}{D-3})$
5	B	0	0	0
5	C	0	0	0
6	A	0	0	0
6	B	0	0	0
6	C	0	0	0
7	A	0	0	$\frac{D(D-1)}{4} - \frac{1}{2}(\frac{D-1}{D-3})$
7	B	0	0	0
7	C	0	0	0
8	A	0	0	0
8	B	0	0	0
8	C	0	0	0

Table 5: Generic contributions from the 3-point vertices.

I	J	$iV_I^{\alpha\beta}(x)\, i[S](x;x')\, iV_J^{\rho\sigma}(x')\, i[{}_{\alpha\beta}\Delta_{\rho\sigma}](x;x')$
1	1	$\frac{1}{4}\kappa^2 \not{\partial}\delta^D(x-x')\, i[{}^{\alpha}{}_{\alpha}\Delta^{\rho}{}_{\rho}](x;x)$
1	2	$-\frac{1}{4}\kappa^2\gamma^\rho\partial^\sigma\delta^D(x-x')\, i[{}^{\alpha}{}_{\alpha}\Delta_{\rho\sigma}](x;x)$
1	3	$\frac{1}{4}i\kappa^2\gamma^\rho J^{\sigma\mu}\delta^D(x-x')\,\partial'_\mu i[{}^{\alpha}{}_{\alpha}\Delta_{\rho\sigma}](x;x')$
2	1	$\frac{1}{4}\kappa^2\partial'_\mu\{\gamma^\alpha\partial^\beta\, i[S](x;x')\,\gamma^\mu\, i[{}_{\alpha\beta}\Delta^{\rho}{}_{\rho}](x;x')\}$
2	2	$-\frac{1}{4}\kappa^2\partial'^\rho\{\gamma^\alpha\partial^\beta\, i[S](x;x')\,\gamma^\sigma\, i[{}_{\alpha\beta}\Delta_{\rho\sigma}](x;x')\}$
2	3	$-\frac{1}{4}i\kappa^2\,\gamma^\alpha\partial^\beta\, i[S](x;x')\,\gamma^\rho J^{\sigma\mu}\partial'_\mu\, i[{}_{\alpha\beta}\Delta_{\rho\sigma}](x;x')$
3	1	$-\frac{1}{4}i\kappa^2\partial'_\nu\{\gamma^\alpha J^{\beta\mu}\, i[S](x;x')\,\gamma^\nu\partial_\mu\, i[{}_{\alpha\beta}\Delta^{\rho}{}_{\rho}](x;x')\}$
3	2	$\frac{1}{4}i\kappa^2\partial'^\rho\{\gamma^\alpha J^{\beta\mu}\, i[S](x;x')\,\gamma^\sigma\partial_\mu\, i[{}_{\alpha\beta}\Delta_{\rho\sigma}](x;x')\}$
3	3	$-\frac{1}{4}\kappa^2\,\gamma^\alpha J^{\beta\mu}\, i[S](x;x')\,\gamma^\rho J^{\sigma\nu}\partial_\mu\partial'_\nu\, i[{}_{\alpha\beta}\Delta_{\rho\sigma}](x;x')$

Because there are three 3-point vertex operators of Equation 53, there are nine vertex products in Equation 120. We label each contribution by the numbers on its vertex pair, for example,

$$\Bigl[I\text{–}J\Bigr] \equiv iV_I^{\alpha\beta}(x)\times i\Bigl[S\Bigr](x;x')\times iV_J^{\rho\sigma}(x')\times i\Bigl[{}_{\alpha\beta}\Delta_{\rho\sigma}\Bigr](x;x')\;. \qquad (121)$$

Table 5 gives the generic reductions, before decomposing the graviton propagator. Most of these reductions are straightforward but two subtleties deserve mention. First, the Dirac slash of the fermion propagator gives a delta function,

$$i\not{\partial} i\Bigl[S\Bigr](x;x') = i\delta^D(x-x')\;. \qquad (122)$$

This occurs whenever the first vertex is $I=1$, for example,

$$\Bigl[1\text{–}3\Bigr] \;\equiv\; \frac{i\kappa}{2}\eta^{\alpha\beta}i\not{\partial}\times i\Bigl[S\Bigr](x;x')\times -\frac{i\kappa}{2}\gamma^\rho J^{\sigma\mu}\partial'_\mu\times i\Bigl[{}_{\alpha\beta}\Delta_{\rho\sigma}\Bigr](x;x')\;, \qquad (123)$$

$$= \frac{i\kappa^2}{4}\gamma^\rho J^{\sigma\mu}\delta^D(x-x')\,\partial'_\mu i\Bigl[{}^{\alpha}{}_{\alpha}\Delta_{\rho\sigma}\Bigr](x;x')\;. \qquad (124)$$

The second subtlety is that derivatives on external lines must be partially integrated back on the entire diagram. This happens whenever the second vertex is $J=1$ or $J=2$, for example,

$$\Bigl[2\text{–}2\Bigr] \equiv -\frac{i\kappa}{2}\gamma^\alpha i\partial^\beta\times i\Bigl[S\Bigr](x;x')\times -\frac{i\kappa}{2}\gamma^\rho i\partial'^\sigma_{\rm ext}\times i\Bigl[{}_{\alpha\beta}\Delta_{\rho\sigma}\Bigr](x;x')\;, \qquad (125)$$

$$\longrightarrow -\frac{\kappa^2}{4}\partial'^{\sigma}\Big\{\gamma^\alpha\partial^\beta\, i\Big[S\Big](x;x')\,\gamma^\rho\, i\Big[{}_{\alpha\beta}\Delta_{\rho\sigma}\Big](x;x')\Big\}. \tag{126}$$

In comparing Table 5 and Table 2 it will be seen that the 3-point contributions with $I=1$ are closely related to three of the 4-point contributions. In fact the [1–1] contribution is -2 times the 4-point contribution with $I=1$; while [1–2] and [1–3] cancel the 4-point contributions with $I=3$ and $I=5$, respectively. Because of this it is convenient to add the 3-point contributions with $I=1$ to the 4-point contributions from Table 4,

$$-i\Big[\Sigma^{\rm 4pt}+\Sigma^{\rm 3pt}_{I=1}\Big](x;x') = \frac{\kappa^2 H^{D-2}}{(4\pi)^{\frac{D}{2}}}\frac{\Gamma(D-1)}{\Gamma(\frac{D}{2})}\Big\{\Big[-\frac{(D+1)(D-1)(D-4)}{2(D-3)}A \\ -\frac{(D-1)(D^3-8D^2+23D-32)}{8(D-2)^2(D-3)^2}\Big]\bar{\partial\!\!\!/} + \Big[\frac{3}{4}\Big(D-\frac{2}{D-3}\Big)A \\ +\frac{3(D^2-6D+8)}{4(D-2)^2(D-3)^2}-\frac{3}{8}\Big]\overline{\partial\!\!\!/} + \Big(\frac{D-1}{4}\Big)\Big(D-\frac{2}{D-3}\Big)aH\gamma^0\Big\}\delta^D(x-x'). \tag{127}$$

In what follows we will focus on the 3-point contributions with $I=2$ and $I=3$.

3.3 Conformal Contributions

The key to achieving a tractable reduction of the diagrams of Fig. 2 is that the first term of each of the scalar propagators $i\Delta_I(x;x')$ is the conformal propagator $i\Delta_{\rm cf}(x;x)$. The sum of the three index factors also gives a simple tensor, so it is very efficient to write the graviton propagator in the form,

$$i\Big[{}_{\mu\nu}\Delta_{\rho\sigma}\Big](x;x') = \Big[2\eta_{\mu(\rho}\eta_{\sigma)\nu}-\frac{2}{D-2}\eta_{\mu\nu}\eta_{\rho\sigma}\Big]i\Delta_{\rm cf}(x;x') \\ +\sum_{I=A,B,C}\Big[{}_{\mu\nu}T^I_{\rho\sigma}\Big]\, i\delta\Delta_I(x;x')\ , \tag{128}$$

where $i\delta\Delta_I(x;x') \equiv i\Delta_I(x;x')-i\Delta_{\rm cf}(x;x')$. In this subsection we evaluate the contribution to Equation 120 using the 3-point vertex operators of Equation 53 and the fermion propagator of Equation 51 but only the conformal part of the graviton propagator,

$$i\Big[{}_{\mu\nu}\Delta_{\rho\sigma}\Big](x;x') \longrightarrow \Big[2\eta_{\mu(\rho}\eta_{\sigma)\nu}-\frac{2}{D-2}\eta_{\mu\nu}\eta_{\rho\sigma}\Big]i\Delta_{\rm cf}(x;x') \equiv \Big[{}_{\alpha\beta}T^{\rm cf}_{\rho\sigma}\Big]i\Delta_{\rm cf}(x;x)\ . \tag{129}$$

Table 6: Contractions from the $i\Delta_{\rm cf}$ part of the graviton propagator.

I	J	sub	$iV_I^{\alpha\beta}(x)\, i[S](x;x')\, iV_J^{\rho\sigma}(x')\, [{}_{\alpha\beta}T^{\rm cf}_{\rho\sigma}]\, i\Delta_{\rm cf}(x;x')$
2	1		$-\frac{1}{D-2}\kappa^2 \not{\partial}'\{\delta^D(x-x')\, i\Delta_{\rm cf}(x;x)\}$
2	2	a	$-\frac{1}{4}(\frac{D-4}{D-2})\kappa^2 \not{\partial}'\{\delta^D(x-x')\, i\Delta_{\rm cf}(x;x)\}$
2	2	b	$-(\frac{D-2}{4})\kappa^2 \partial'_\mu\{\partial^\mu i[S](x;x')\, i\Delta_{\rm cf}(x;x')\}$
2	3	a	$\frac{1}{8}(\frac{D}{D-2})\kappa^2 \delta^D(x-x')\not{\partial}'\, i\Delta_{\rm cf}(x;x)$
2	3	b	$+(\frac{D-2}{8})\kappa^2 \partial_\mu\, i[S](x;x')\partial'^\mu\, i\Delta_{\rm cf}(x;x')$
3	1		$\frac{1}{2}(\frac{D-1}{D-2})\kappa^2 \partial'_\mu\{\not{\partial}\, i\Delta_{\rm cf}(x;x)\, i[S](x;x')\gamma^\mu\}$
3	2	a	$-\frac{1}{4(D-2)}\kappa^2 \partial'_\mu\{\not{\partial}\, i\Delta_{\rm cf}(x;x)\, i[S](x;x')\gamma^\mu\}$
3	2	b	$-(\frac{D-2}{8})\kappa^2 \partial'_\mu\{i[S](x;x')\, \partial^\mu i\Delta_{\rm cf}(x;x)\}$
3	2	c	$-\frac{1}{8}\kappa^2 \not{\partial}'\{i[S](x;x')\, \not{\partial}\, i\Delta_{\rm cf}(x;x)\}$
3	3	a	$(\frac{D-2}{16})\kappa^2 i[S](x;x')\partial\cdot\partial' i\Delta_{\rm cf}(x;x')$
3	3	b	$-\frac{1}{8}(\frac{2D-3}{D-2})\kappa^2 \gamma^\mu i[S](x;x')\partial_\mu \not{\partial}'\, i\Delta_{\rm cf}(x;x)$
3	3	c	$+\frac{1}{16}\kappa^2 \gamma^\mu i[S](x;x')\partial'_\mu \not{\partial}\, i\Delta_{\rm cf}(x;x)$

We carry out the reduction in three stages. In the first stage the conformal part 129 of the graviton propagator is substituted into the generic results from Table 5 and the contractions are performed. We also make use of gamma matrix identities such as Equation 32 and,

$$\gamma^\mu i\Bigl[S\Bigr](x;x')\gamma_\mu = (D-2) i\Bigl[S\Bigr](x;x') \qquad \text{and} \qquad \gamma_\alpha J^{\alpha\mu} = -\frac{i}{2}(D-1)\gamma^\mu \; . \tag{130}$$

Finally, we employ relation 122 whenever $\not{\partial}'$ acts upon the fermion propagator. However, we do not at this stage act any other derivatives. The results of these reductions are summarized in Table 6. Because the conformal tensor factor $[{}_{\alpha\beta}T^{\rm cf}_{\rho\sigma}]$ contains three distinct terms, and because the factors of $\gamma^\alpha J^{\beta\mu}$ in Table 5 can contribute different terms with a distinct structure, we have sometimes broken up the result for a given vertex pair into parts. These parts are distinguished in Table 6 and subsequently by subscripts taken from the lower case Latin letters.

In the second stage we substitute the fermion and conformal propagators,

$$i\Big[S\Big](x;x') = -\frac{i\Gamma(\frac{D}{2})}{2\pi^{\frac{D}{2}}} \frac{\gamma^{\mu}\Delta x_{\mu}}{\Delta x^{D}} \; , \tag{131}$$

$$i\Delta_{\rm cf}(x;x') = \frac{\Gamma(\frac{D}{2}-1)}{4\pi^{\frac{D}{2}}} \frac{(aa')^{1-\frac{D}{2}}}{\Delta x^{D-2}} \; . \tag{132}$$

At this stage we take advantage of the curious consequence of the automatic subtraction of dimension regularization that any dimension-dependent power of zero is discarded,

$$\lim_{x'\to x} i\Delta_{\rm cf}(x;x') = 0 \qquad \text{and} \qquad \lim_{x'\to x} \partial'_{\mu} i\Delta_{\rm cf}(x;x') = 0 \; . \tag{133}$$

In the final stage we act the derivatives. These can act upon the conformal coordinate separation $\Delta x^{\mu} \equiv x^{\mu} - x'^{\mu}$, or upon the factor of $(aa')^{1-\frac{D}{2}}$ from the conformal propagator. We quote separate results for the cases where all derivatives act upon the conformal coordinate separation (Table 7) and the case where one or more of the derivatives acts upon the scale factors (Table 8). In the former case the final result must in each case take the form of a pure number times the universal factor,

$$\frac{(aa')^{1-\frac{D}{2}} \gamma^{\mu}\Delta x_{\mu}}{\Delta x^{2D}} \; . \tag{134}$$

The sum of all terms in Table 7 is,

$$-i\Big[\Sigma^{T7}\Big](x;x') = \frac{i\kappa^2}{2^6\pi^D} \Gamma\Big(\frac{D}{2}\Big) \Gamma\Big(\frac{D}{2}-1\Big) (-2D^2+5D-4)(D-1)(aa')^{1-\frac{D}{2}} \frac{\gamma^{\mu}\Delta x_{\mu}}{\Delta x^{2D}} \; . \tag{135}$$

If one simply omits the factor of $(aa')^{1-\frac{D}{2}}$ the result is the same as in flat space. Although Equation 135 is well defined for $x'^{\mu} \neq x^{\mu}$ we must remember that $[\Sigma](x;x')$ will be used inside an integral in the quantum-corrected Dirac equation shown by Equation 24. For that purpose the singularity at $x'^{\mu} = x^{\mu}$ is cubicly divergent in $D=4$ dimensions. To renormalize this divergence we extract derivatives with respect to the coordinate x^{μ}, which can of course be taken outside the integral in Equation 24 to give a less singular integrand,

$$\frac{\gamma^{\mu}\Delta x_{\mu}}{\Delta x^{2D}} = \frac{-\not{\partial}}{2(D-1)} \left\{ \frac{1}{\Delta x^{2D-2}} \right\} , \tag{136}$$

Table 7: Conformal $i\Delta_{\rm cf}$ terms in which all derivatives act upon $\Delta x^2(x;x')$. All contributions are multiplied by $\frac{i\kappa^2}{8\pi^D}\Gamma(\frac{D}{2})\Gamma(\frac{D}{2}-1)(aa')^{1-\frac{D}{2}}$.

I	J	sub	Coefficient of $\frac{\gamma^\mu \Delta x_\mu}{\Delta x^{2D}}$
2	1		0
2	2	a	0
2	2	b	$-\frac{1}{4}(D-2)^2(D-1)$
2	3	a	0
2	3	b	$\frac{1}{8}(D-2)^2(D-1)$
3	1		$-(D-1)^2$
3	2	a	$\frac{1}{2}(D-1)$
3	2	b	$-\frac{1}{8}(D-2)^2(D-1)$
3	2	c	$\frac{1}{4}(D-2)(D-1)$
3	3	a	0
3	3	b	$\frac{1}{4}(2D-3)(D-1)$
3	3	c	$-\frac{1}{8}(D-2)(D-1)$

$$= \frac{-\not{\partial}\,\partial^2}{4(D-1)(D-2)^2}\Bigl(\frac{1}{\Delta x^{2D-4}}\Bigr) , \tag{137}$$

$$= \frac{-\not{\partial}\,\partial^4}{8(D-1)(D-2)^2(D-3)(D-4)}\Bigl(\frac{1}{\Delta x^{2D-6}}\Bigr) . \tag{138}$$

Expression 138 is integrable in four dimensions and we could take $D=4$ except for the explicit factor of $1/(D-4)$. Of course that is how ultraviolet divergences manifest in dimensional regularization. We can segregate the divergence on a local term by employing a simple representation for a delta function,

$$\frac{\partial^2}{D-4}\Bigl(\frac{1}{\Delta x^{2D-6}}\Bigr) = \frac{\partial^2}{D-4}\Bigl\{\frac{1}{\Delta x^{2D-6}} - \frac{\mu^{D-4}}{\Delta x^{D-2}}\Bigr\} + \frac{i4\pi^{\frac{D}{2}}\mu^{D-4}}{\Gamma(\frac{D}{2}-1)}\frac{\delta^D(x-x')}{D-4}, \tag{139}$$

$$= -\frac{\partial^2}{2}\Bigl\{\frac{\ln(\mu^2\Delta x^2)}{\Delta x^2} + O(D-4)\Bigr\} + \frac{i4\pi^{\frac{D}{2}}\mu^{D-4}}{\Gamma(\frac{D}{2}-1)}\frac{\delta^D(x-x')}{D-4}. \tag{140}$$

The final result for Table 7 is,

$$-i\Big[\Sigma^{T7}\Big](x;x') = -\frac{i\kappa^2}{2^8\pi^4}\frac{1}{aa'}\,\partial\!\!\!/\,\partial^4\Bigg\{\frac{\ln(\mu^2\Delta x^2)}{\Delta x^2}\Bigg\} + O(D\!-\!4) \\ -\frac{\kappa^2\mu^{D-4}}{2^8\pi^{\frac{D}{2}}}\Gamma\Big(\frac{D}{2}\!-\!1\Big)\frac{(2D^2\!-\!5D\!+\!4)(aa')^{1-\frac{D}{2}}}{(D\!-\!2)(D\!-\!3)(D\!-\!4)}\,\partial\!\!\!/\,\partial^2\delta^D(x\!-\!x'). \quad (141)$$

When one or more derivative acts upon the scale factors a bewildering variety of spacetime and gamma matrix structures result. For example, the $[3–2]_b$ term gives,

$$-\Big(\frac{D\!-\!2}{8}\Big)\kappa^2\partial'_\mu\Bigg\{i\Big[S\Big](x;x')\partial^\mu i\Delta_{\rm cf}(x;x')\Bigg\} \\ = \frac{i\kappa^2}{32\pi^D}\Gamma^2\Big(\frac{D}{2}\Big)\partial'_\mu\Bigg\{\frac{\gamma^\nu\Delta x_\nu}{\Delta x^D}(aa')^{1-\frac{D}{2}}\Bigg[-\frac{(D\!-\!2)\Delta x^\mu}{\Delta x^D} + \frac{(D\!-\!2)Ha\delta^\mu_0}{2\Delta x^{D-2}}\Bigg]\Bigg\}, \quad (142)$$

$$= \frac{i\kappa^2}{32\pi^D}\Gamma^2\Big(\frac{D}{2}\Big)(aa')^{1-\frac{D}{2}}\Bigg\{-\frac{(D\!-\!1)(D\!-\!2)\gamma^\mu\Delta x_\mu}{\Delta x^{2D}} + \frac{(D\!-\!2)Ha\gamma^0}{2\Delta x^{2D-2}} \\ +\frac{(D\!-\!2)^2a'H\Delta\eta\gamma^\mu\Delta x_\mu}{2\Delta x^{2D}} - \frac{(D\!-\!1)(D\!-\!2)aH\Delta\eta\gamma^\mu\Delta x_\mu}{\Delta x^{2D}} \\ -\frac{(D\!-\!2)^2aa'H^2\gamma^\mu\Delta x_\mu}{4\Delta x^{2D-2}}\Bigg\}. \quad (143)$$

The first term of Equation 143 originates from both derivatives acting on the conformal coordinate separation. It belongs in Table 7. The next three terms come from a single derivative acting on a scale factor, and the final term in Equation 143 derives from both derivatives acting upon scale factors. These last four terms belong in Table 8. They can be expressed as dimensionless functions of D, a and a' times three basic terms,

$$\frac{i\kappa^2}{16\pi^D}\Gamma^2\Big(\frac{D}{2}\Big)(aa')^{1-\frac{D}{2}}\Bigg\{-\frac{1}{8}(D\!-\!2)^2\times\frac{aa'H^2\gamma^\mu\Delta x_\mu}{\Delta x^{2D-2}} + \frac{1}{4}(D\!-\!2)a\times\frac{H\gamma^0}{\Delta x^{2D-2}} \\ +\Big[\frac{1}{4}(D\!-\!2)^2a' - \frac{1}{2}(D\!-\!1)(D\!-\!2)a\Big]\times\frac{H\Delta\eta\gamma^\mu\Delta x_\mu}{\Delta x^{2D}}\Bigg\}. \quad (144)$$

These three terms turn out to be all we need, although intermediate expressions sometimes show other kinds. An example is the [3–1] term,

$$\frac{1}{2}\Big(\frac{D\!-\!1}{D\!-\!2}\Big)\kappa^2\partial'_\mu\Bigg\{\partial\!\!\!/\, i\Delta_{\rm cf}(x;x)\, i\Big[S\Big](x;x')\gamma^\mu\Bigg\}$$

Table 8: Conformal $i\Delta_{\rm cf}$ terms in which some derivatives act upon scale factors. All contributions are multiplied by $\frac{i\kappa^2}{16\pi^D}\,\Gamma^2(\frac{D}{2})(aa')^{1-\frac{D}{2}}$.

I	J	sub	$\frac{aa'H^2\gamma^\mu\Delta x_\mu}{\Delta x^{2D-2}}$	$\frac{H\gamma^0}{\Delta x^{2D-2}}$	$\frac{H\Delta\eta\gamma^\mu\Delta x_\mu}{\Delta x^{2D}}$
2	1		0	0	0
2	2	a	0	0	0
2	2	b	0	$-\frac{1}{2}(D\!-\!2)a'$	$\frac{1}{2}(D\!-\!2)Da'$
2	3	a	0	0	0
2	3	b	0	$\frac{1}{4}(D\!-\!2)a'$	$-\frac{1}{4}(D\!-\!2)Da'$
3	1		$\frac{1}{2}(D\!-\!1)$	0	0
3	2	a	$-\frac{1}{4}$	0	0
3	2	b	$-\frac{1}{8}(D\!-\!2)^2$	$\frac{1}{4}(D\!-\!2)a$	$\frac{1}{4}(D\!-\!2)^2a'$ $-\frac{1}{2}(D\!-\!2)(D\!-\!1)a$
3	2	c	$-\frac{1}{8}(D\!-\!2)$	0	0
3	3	a	$\frac{1}{16}(D\!-\!2)^2$	0	$\frac{1}{8}(D\!-\!2)^2(a\!-\!a')$
3	3	b	$-\frac{1}{8}(2D\!-\!3)$	0	0
3	3	c	$\frac{1}{16}(D\!-\!2)$	0	0

$$= \frac{i\kappa^2}{8\pi^D}\Gamma^2\Bigl(\frac{D}{2}\Bigr)\Bigl(\frac{D\!-\!1}{D\!-\!2}\Bigr)\partial'_\mu\Biggl\{(aa')^{1-\frac{D}{2}}\Biggl[\frac{\gamma^\alpha\Delta x_\alpha}{\Delta x^D}+\frac{aH\gamma^0}{2\Delta x^{D-2}}\Biggr]\frac{\gamma^\beta\Delta x_\beta}{\Delta x^D}\gamma^\mu\Biggr\}, \quad (145)$$

$$= \frac{i\kappa^2}{8\pi^D}\Gamma^2\Bigl(\frac{D}{2}\Bigr)(aa')^{1-\frac{D}{2}}\Biggl\{-2\frac{(D\!-\!1)^2}{(D\!-\!2)}\frac{\gamma^\mu\Delta x_\mu}{\Delta x^{2D}}-\frac{1}{2}(D\!-\!1)\frac{aH\gamma^0}{\Delta x^{2D-2}}$$
$$+\frac{1}{2}(D\!-\!1)\frac{a'H\gamma^0}{\Delta x^{2D-2}}-\frac{1}{4}(D\!-\!1)\frac{aa'H^2\gamma^0\gamma^\mu\Delta x_\mu\gamma^0}{\Delta x^{2D-2}}\Biggr\}. \quad (146)$$

As before, the first term in Equation 146 belongs in Table 7. The second and third terms are of a type we encountered in Equation 143 but the final term is not. However, it is simple to bring this term to standard form by anti-commuting the γ^μ through either γ^0,

$$aa'H^2\gamma^0\gamma^\mu\Delta x_\mu\gamma^0 = -aa'H^2\gamma^\mu\Delta x_\mu - 2aa'H^2\Delta\eta\gamma^0\,, \quad (147)$$
$$= -aa'H^2\gamma^\mu\Delta x_\mu - 2(a\!-\!a')H\gamma^0\,. \quad (148)$$

Note our use of the identity $(a\!-\!a') = aa'H\Delta\eta$.

When all terms in Table 8 are summed it emerges that a factor of $H^2 a a'$ can be extracted,

$$
\begin{aligned}
-i\Big[\Sigma^{T8}\Big](x;x') &= \frac{i\kappa^2}{16\pi^D}\Gamma^2\Big(\frac{D}{2}\Big)(aa')^{1-\frac{D}{2}}\Bigg\{-\frac{1}{16}(D^2-7D+8)\times\frac{aa'H^2\gamma^\mu\Delta x_\mu}{\Delta x^{2D-2}} \\
&\quad +\frac{1}{4}(D-2)(a-a')\times\frac{H\gamma^0}{\Delta x^{2D-2}}-\frac{1}{8}(D-2)(3D-2)(a-a')\times\frac{H\Delta\eta\gamma^\mu\Delta x_\mu}{\Delta x^{2D}}\Bigg\}, \quad (149) \\
&= \frac{i\kappa^2 H^2}{16\pi^D}\Gamma^2\Big(\frac{D}{2}\Big)(aa')^{2-\frac{D}{2}}\Bigg\{-\frac{1}{16}(D^2-7D+8)\times\frac{\gamma^\mu\Delta x_\mu}{\Delta x^{2D-2}} \\
&\quad +\frac{1}{4}(D-2)\times\frac{\gamma^0\Delta\eta}{\Delta x^{2D-2}}-\frac{1}{8}(D-2)(3D-2)\times\frac{\Delta\eta^2\gamma^\mu\Delta x_\mu}{\Delta x^{2D}}\Bigg\}. \quad (150)
\end{aligned}
$$

Note the fact that this expression is odd under interchange of x^μ and x'^μ. Although individual contributions to the last two columns of Table 8 are not odd under interchange, their sum always produces a factor of $a-a' = aa'H\Delta\eta$ which makes Equation 150 odd.

Expression 150 can be simplified using the differential identities,

$$
\begin{aligned}
\frac{\Delta\eta^2\gamma^\mu\Delta x_\mu}{\Delta x^{2D}} &= \frac{\partial_0^2}{4(D-2)(D-1)}\Big(\frac{\gamma^\mu\Delta x_\mu}{\Delta x^{2D-4}}\Big) \\
&\quad -\frac{1}{2(D-1)}\,\frac{\gamma^\mu\Delta x_\mu}{\Delta x^{2D-2}}+\frac{1}{D-1}\,\frac{\gamma^0\Delta\eta}{\Delta x^{2D-2}}\;, \quad (151) \\
\frac{\gamma^0\Delta\eta}{\Delta x^{2D-2}} &= \frac{\gamma^0\partial_0}{2(D-2)}\Big(\frac{1}{\Delta x^{2D-4}}\Big)\;. \quad (152)
\end{aligned}
$$

The result is,

$$
\begin{aligned}
-i\Big[\Sigma^{T8}\Big](x;x') &= \frac{i\kappa^2 H^2}{16\pi^D}\Gamma^2\Big(\frac{D}{2}\Big)(aa')^{2-\frac{D}{2}}\Bigg\{-\frac{(D^3-11D^2+23D-12)}{16(D-1)}\,\frac{\gamma^\mu\Delta x_\mu}{\Delta x^{2D-2}} \\
&\quad -\frac{D}{16(D-1)}\,\gamma^0\partial_0\Big(\frac{1}{\Delta x^{2D-4}}\Big)-\frac{1}{32}\Big(\frac{3D-2}{D-1}\Big)\partial_0^2\Big(\frac{\gamma^\mu\Delta x_\mu}{\Delta x^{2D-4}}\Big)\Bigg\}. \quad (153)
\end{aligned}
$$

We now exploit partial integration identities of the same type as those previously used for Table 7,

$$
\begin{aligned}
\frac{\gamma^\mu\Delta x_\mu}{\Delta x^{2D-4}} &= \frac{-\partial\!\!\!/}{2(D-3)}\Big(\frac{1}{\Delta x^{2D-6}}\Big) = -\frac{\partial\!\!\!/}{2}\Big(\frac{1}{\Delta x^2}\Big)+O(D-4)\;, \quad (154) \\
\frac{\gamma^\mu\Delta x_\mu}{\Delta x^{2D-2}} &= \frac{-\partial\!\!\!/\,\partial^2}{4(D-2)(D-3)(D-4)}\Big(\frac{1}{\Delta x^{2D-6}}\Big)\;,
\end{aligned}
$$

$$= \frac{\not{\partial}\,\partial^2}{16}\Bigl\{\frac{\ln(\mu^2\Delta x^2)}{\Delta x^2}\Bigr\} + O(D-4) - \frac{i\pi^{\frac{D}{2}}\mu^{D-4}}{2\Gamma(\frac{D}{2})}\,\frac{\not{\partial}\,\delta^D(x-x')}{(D-3)(D-4)}\;, \qquad (155)$$

$$\frac{1}{\Delta x^{2D-4}} = \frac{\partial^2}{2(D-3)(D-4)}\Bigl(\frac{1}{\Delta x^{2D-6}}\Bigr)\;,$$

$$= -\frac{\partial^2}{4}\Bigl\{\frac{\ln(\mu^2\Delta x^2)}{\Delta x^2}\Bigr\} + O(D-4) + \frac{i2\pi^{\frac{D}{2}}\mu^{D-4}}{\Gamma(\frac{D}{2}-1)}\,\frac{\delta^D(x-x')}{(D-3)(D-4)}\;. \qquad (156)$$

It is also useful to convert temporal derivatives to spatial ones using,

$$\gamma^0\partial_0 = \not{\partial} - \overline{\not{\partial}} \qquad \text{and} \qquad \partial_0^2 = \nabla^2 - \partial^2\;. \qquad (157)$$

Substituting these relations in Equation 153 gives,

$$-i\Bigl[\Sigma^{T8}\Bigr](x;x') = \frac{\kappa^2 H^2\mu^{D-4}\Gamma(\frac{D}{2})\,(aa')^{2-\frac{D}{2}}}{2^9\pi^{\frac{D}{2}}(D-1)(D-3)(D-4)}\Bigl\{-\Bigl(D^3-13D^2+27D-12\Bigr)\not{\partial} \\ -2D(D-2)\overline{\not{\partial}}\Bigr\}\delta^D(x-x') + \frac{i\kappa^2H^2}{2^9\cdot 3\cdot\pi^4}\Bigl\{\Bigl[6\,\not{\partial}\,\partial^2 - 2\overline{\not{\partial}}\,\partial^2\Bigr]\Bigl(\frac{\ln(\mu^2\Delta x^2)}{\Delta x^2}\Bigr) \\ +5\,\not{\partial}\,(\nabla^2-\partial^2)\Bigl(\frac{1}{\Delta x^2}\Bigr)\Bigr\} + O(D-4)\;. \qquad (158)$$

3.4 Sub-Leading Contributions from $i\delta\Delta_A$

In this subsection we work out the contribution from substituting the residual A-type part of the graviton propagator in Table 5,

$$i\Bigl[{}_{\alpha\beta}\Delta_{\rho\sigma}\Bigr](x;x') \longrightarrow \Bigl[\overline{\eta}_{\alpha\rho}\overline{\eta}_{\sigma\beta} + \overline{\eta}_{\alpha\sigma}\overline{\eta}_{\rho\beta} - \frac{2}{D-3}\overline{\eta}_{\alpha\beta}\overline{\eta}_{\rho\sigma}\Bigr] i\delta\Delta_A(x;x')\;. \qquad (159)$$

As with the conformal contributions of the previous section we first make the requisite contractions and then act the derivatives. The result of this first step is summarized in Table 9. We have sometimes broken the result for a single vertex pair into as many as five terms because the three different tensors in Equation 159 can make distinct contributions, and because distinct contributions also come from breaking up factors of $\gamma^\alpha J^{\beta\mu}$. These distinct contributions are labeled by subscripts a, b, c, etc. We have tried to arrange them so that terms closer to the beginning of the alphabet have fewer purely spatial derivatives.

Table 9: Contractions from the $i\delta\Delta_A$ part of the graviton propagator

I	J	sub	$iV_I^{\alpha\beta}(x)\, i[S](x;x')\, iV_J^{\rho\sigma}(x')\, [{}_{\alpha\beta}T^A_{\rho\sigma}]\, i\delta\Delta_A(x;x')$
2	1		$-\frac{1}{(D-3)}\kappa^2\partial'_\mu\{\bar{\partial\!\!\!/}i[S](x;x')\gamma^\mu i\delta\Delta_A(x;x')\}$
2	2	a	$\frac{1}{4}\kappa^2\,\bar{\partial\!\!\!/}\{\partial_k i[S](x;x')\gamma_k i\delta\Delta_A(x;x')\}$
2	2	b	$+\frac{1}{4}\kappa^2\partial_\ell\{\gamma_k\partial_\ell i[S](x;x')\gamma_k i\delta\Delta_A(x;x')\}$
2	2	c	$-\frac{1}{2(D-3)}\kappa^2\partial_k\{\bar{\partial\!\!\!/}i[S](x;x')\gamma_k i\delta\Delta_A(x;x')\}$
2	3	a	$\frac{1}{2(D-3)}\kappa^2\,\bar{\partial\!\!\!/}i[S](x;x')\partial\!\!\!/' i\delta\Delta_A(x;x')$
2	3	b	$-\frac{1}{4}\kappa^2\gamma_k\partial_\ell i[S](x;x')\gamma_{(k}\partial_{\ell)}i\delta\Delta_A(x;x')$
2	3	c	$+\frac{1}{4(D-3)}\kappa^2\,\bar{\partial\!\!\!/}i[S](x;x')\,\bar{\partial\!\!\!/}i\delta\Delta_A(x;x')$
3	1	a	$\frac{1}{2}(\frac{D-1}{D-3})\kappa^2\partial'_\mu\{\partial\!\!\!/ i\delta\Delta_A(x;x')i[S](x;x')\gamma^\mu\}$
3	1	b	$-\frac{1}{2(D-3)}\kappa^2\partial'_\mu\{\bar{\partial\!\!\!/}i\delta\Delta_A(x;x')i[S](x;x')\gamma^\mu\}$
3	2	a	$\frac{1}{2(D-3)}\kappa^2\partial_k\{\partial\!\!\!/ i\delta\Delta_A(x;x')i[S](x;x')\gamma_k\}$
3	2	b	$-\frac{1}{4(D-3)}\kappa^2\partial_k\{\bar{\partial\!\!\!/}i\delta\Delta_A(x;x')i[S](x;x')\gamma_k\}$
3	2	c	$+\frac{1}{8}\kappa^2\,\bar{\partial\!\!\!/}\{i[S](x;x')\,\bar{\partial\!\!\!/}i\delta\Delta_A(x;x')\}$
3	2	d	$+\frac{1}{8}\kappa^2\partial_k\{\gamma_\ell i[S](x;x')\gamma_\ell\partial_k i\delta\Delta_A(x;x')\}$
3	3	a	$-\frac{1}{4}(\frac{D-1}{D-3})\kappa^2\gamma^\mu i[S](x;x')\partial_\mu\partial\!\!\!/' i\delta\Delta_A(x;x')$
3	3	b	$-\frac{1}{4(D-3)}\kappa^2\gamma^\mu i[S](x;x')\partial_\mu\,\bar{\partial\!\!\!/}i\delta\Delta_A(x;x')$
3	3	c	$+\frac{1}{4(D-3)}\kappa^2\gamma_k i[S](x;x')\partial_k\partial\!\!\!/' i\delta\Delta_A(x;x')$
3	3	d	$-\frac{1}{16}(\frac{D-5}{D-3})\kappa^2\gamma_k i[S](x;x')\partial_k\,\bar{\partial\!\!\!/}i\delta\Delta_A(x;x')$
3	3	e	$-\frac{1}{16}\kappa^2\gamma_k i[S](x;x')\gamma_k\nabla^2 i\delta\Delta_A(x;x')$

The next step is to act the derivatives and it is of course necessary to have an expression for $i\delta\Delta_A(x;x')$ at this stage. From Equation 83 one can infer,

$$i\delta\Delta_A(x;x') = \frac{H^2}{16\pi^{\frac{D}{2}}}\frac{\Gamma(\frac{D}{2}+1)}{\frac{D}{2}-2}\frac{(aa')^{2-\frac{D}{2}}}{\Delta x^{D-4}} + \frac{H^{D-2}}{(4\pi)^{\frac{D}{2}}}\frac{\Gamma(D-1)}{\Gamma(\frac{D}{2})}\Big\{-\pi\cot\Big(\frac{\pi}{2}D\Big) + \ln(aa')\Big\}$$

Table 10: Residual $i\delta\Delta_A$ terms giving both powers of Δx^2. The two coefficients are $A_1 \equiv \frac{i\kappa^2 H^2}{2^6\pi^D}\Gamma(\frac{D}{2}+1)\times\ \Gamma(\frac{D}{2})(aa')^{2-\frac{D}{2}}$ and $A_2 \equiv \frac{i\kappa^2 H^{D-2}}{2^{D+2}\pi^D}\Gamma\big(D-2\big)\big[\ln(aa')-\pi\cot\big(\frac{D\pi}{2}\big)\big]$.

Function	Vertex Pair 2-1	Vertex Pair 2-2
$A_1\partial^2\,\partial\!\!\!/(\frac{1}{\Delta x^{2D-6}})$	$\frac{(D-1)}{(D-2)(D-3)^2(D-4)}$	0
$A_1\partial^2\,\overline{\partial\!\!\!/}(\frac{1}{\Delta x^{2D-6}})$	$\frac{-D}{(D-2)(D-3)^2(D-4)}$	$\frac{-1}{(D-2)(D-3)^2(D-4)}$
$A_2\partial^2\,\overline{\partial\!\!\!/}(\frac{1}{\Delta x^{D-2}})$	$\frac{-2}{D-3}$	0
$A_1\nabla^2\,\partial\!\!\!/(\frac{1}{\Delta x^{2D-6}})$	0	$\frac{D(D^2-3D-2)}{4(D-2)(D-3)^2(D-4)}$
$A_2\nabla^2\,\partial\!\!\!/(\frac{1}{\Delta x^{D-2}})$	0	$\frac{(D^2-3D-2)}{2(D-3)}$
$A_1\nabla^2\,\overline{\partial\!\!\!/}(\frac{1}{\Delta x^{2D-6}})$	0	$\frac{-D}{(D-2)(D-3)^2}$
$A_2\nabla^2\,\overline{\partial\!\!\!/}(\frac{1}{\Delta x^{D-2}})$	0	$-2(\frac{D-4}{D-3})$

$$+\frac{H^{D-2}}{(4\pi)^{\frac{D}{2}}}\sum_{n=1}^{\infty}\Biggl\{\frac{1}{n}\frac{\Gamma(n+D-1)}{\Gamma(n+\frac{D}{2})}\Bigl(\frac{y}{4}\Bigr)^n - \frac{1}{n-\frac{D}{2}+2}\frac{\Gamma(n+\frac{D}{2}+1)}{\Gamma(n+2)}\Bigl(\frac{y}{4}\Bigr)^{n-\frac{D}{2}+2}\Biggr\}\,. \qquad (160)$$

In $D=4$ the most singular contributions to Equation 120 have the form, $i\delta\Delta_A/\Delta x^5$. Because the infinite series terms in Equation 160 go like positive powers of Δx^2 these terms make integrable contributions to the quantum-corrected Dirac equation in Equation 24. We can therefore take $D=4$ for those terms, at which point all the infinite series terms drop. Hence it is only necessary to keep the first line of Equation 160 and that is all we shall ever use.

The contributions from $i\delta\Delta_A$ are more complicated than those from $i\Delta_{\rm cf}$ for several reasons. The fact that there is a second series in Equation 160 occasions our Table 10. These contributions are distinguished by all derivatives acting upon the conformal coordinate separation and by both series making nonzero contributions. Because these terms are special we shall explicitly carry out the reduction of the 2–2 contribution. All three 2–2 contractions on Table 9 can be expressed as a certain tensor contracted into a generic form,

$$\Bigl[\delta_{ij}\delta_{k\ell}+\delta_{ik}\delta_{j\ell}-\frac{2}{D-3}\delta_{i\ell}\delta_{jk}\Bigr]\times\frac{\kappa^2}{4}\partial_i\gamma_j\Bigl\{i\delta\Delta_A(x;x')\partial_k i[S](x;x')\gamma_\ell\Bigr\}\,. \qquad (161)$$

So we may as well work out the generic term and then do the contractions

at the end. Substituting the fermion propagator brings this generic term to the form,

$$\text{Generic} \equiv \frac{\kappa^2}{4}\partial_i\gamma_j\Big\{i\delta\Delta_A(x;x')\partial_k i[S](x;x')\gamma_\ell\Big\} , \tag{162}$$

$$= -\frac{i\kappa^2\Gamma(\frac{D}{2})}{8\pi^{\frac{D}{2}}}\partial_i\gamma_j\Big\{i\delta\Delta_A(x;x')\partial_k\Big(\frac{\gamma^\mu\Delta x_\mu}{\Delta x^D}\Big)\gamma_\ell\Big\} . \tag{163}$$

Now recall that there are two sorts of terms in the only part of $i\delta\Delta_A(x;x')$ that can make a nonzero contribution for $D\!=\!4$,

$$i\delta\Delta_{A1}(x;x') \equiv \frac{H^2}{16\pi^{\frac{D}{2}}}\frac{\Gamma(\frac{D}{2}\!+\!1)}{\frac{D}{2}\!-\!2}\frac{(aa')^{2-\frac{D}{2}}}{\Delta x^{D-4}} , \tag{164}$$

$$i\delta\Delta_{A2}(x;x') \equiv \frac{H^{D-2}}{(4\pi)^{\frac{D}{2}}}\frac{\Gamma(D\!-\!1)}{\Gamma(\frac{D}{2})}\Big\{-\pi\cot\Big(\frac{\pi}{2}D\Big)+\ln(aa')\Big\} . \tag{165}$$

Because all the derivatives are spatial we can pass the scale factors outside to obtain,

$$\text{Generic}_1$$

$$= -\frac{i\kappa^2H^2}{2^6\pi^D}\frac{\Gamma(\frac{D}{2})\Gamma(\frac{D}{2}\!+\!1)}{(D\!-\!4)}(aa')^{2-\frac{D}{2}}\partial_i\gamma_j\Big\{\frac{1}{\Delta x^{D-4}}\partial_k\Big(\frac{\gamma^\mu\Delta x_\mu}{\Delta x^D}\Big)\gamma_\ell\Big\} , \tag{166}$$

$$\text{Generic}_2$$

$$= -\frac{i\kappa^2H^{D-2}}{2^{D+3}\pi^D}\Gamma(D\!-\!1)\Big\{-\pi\cot\Big(\frac{\pi}{2}D\Big)\!+\!\ln(aa')\Big\}\partial_i\gamma_j\partial_k\Big(\frac{\gamma^\mu\Delta x_\mu}{\Delta x^D}\Big)\gamma_\ell , \tag{167}$$

$$= \frac{i\kappa^2H^{D-2}}{2^{D+3}\pi^D}\Gamma(D\!-\!2)\Big\{-\pi\cot\Big(\frac{\pi}{2}D\Big)\!+\!\ln(aa')\Big\}\partial_i\gamma_j\partial_k\,\not{\partial}\gamma_\ell\Big(\frac{1}{\Delta x^{D-2}}\Big) . \tag{168}$$

To complete the reduction of the first generic term we note,

$$\frac{1}{\Delta x^{D-4}}\partial_k\Big(\frac{\gamma^\mu\Delta x_\mu}{\Delta x^D}\Big) = \frac{\gamma_k}{\Delta x^{2D-4}}-\frac{D\gamma^\mu\Delta x_\mu\Delta x_k}{\Delta x^{2D-2}} , \tag{169}$$

$$= \frac{1}{2}\Big(\frac{D\!-\!4}{D\!-\!2}\Big)\frac{\gamma_k}{\Delta x^{2D-4}}+\frac{D}{2(D\!-\!2)}\partial_k\Big(\frac{\gamma^\mu\Delta x_\mu}{\Delta x^{2D-4}}\Big) , \tag{170}$$

$$= \frac{1}{4(D\!-\!3)(D\!-\!2)}\Big\{\gamma_k\partial^2-D\partial_k\,\not{\partial}\Big\}\frac{1}{\Delta x^{2D-6}} . \tag{171}$$

Hence the first generic term is,

$$\text{Generic}_1 = \frac{i\kappa^2H^2}{2^8\pi^D}\frac{\Gamma(\frac{D}{2})\Gamma(\frac{D}{2}\!+\!1)}{(D\!-\!4)(D\!-\!3)(D\!-\!2)}(aa')^{2-\frac{D}{2}}$$

$$\times\Big\{D\partial_i\gamma_j\partial_k\,\not{\partial}\gamma_\ell-\partial^2\partial_i\gamma_j\gamma_k\gamma_\ell\Big\}\frac{1}{\Delta x^{2D-6}} . \tag{172}$$

Now we contract the tensor prefactor of Equation 161 into the appropriate spinor-differential operators. For the first generic term this is,

$$\Big[\delta_{ij}\delta_{k\ell}+\delta_{ik}\delta_{j\ell}-\frac{2}{D-3}\delta_{i\ell}\delta_{jk}\Big]\times\Big\{D\partial_i\gamma_j\partial_k\,\not{\partial}\gamma_\ell-\partial^2\partial_i\gamma_j\gamma_k\gamma_\ell\Big\}$$
$$=D\Big(\frac{D-5}{D-3}\Big)\,\overline{\not{\partial}}\,\not{\partial}\,\overline{\not{\partial}}+D\nabla^2\gamma_i\,\not{\partial}\gamma_i-\partial^2\,\overline{\not{\partial}}\gamma_i\gamma_i-\partial^2\gamma_i\,\overline{\not{\partial}}\gamma_i+\frac{2}{D-3}\partial^2\gamma_i\gamma_i\,\overline{\not{\partial}}\;. \quad (173)$$

This term can be simplified using the identities,

$$\begin{aligned}
\overline{\not{\partial}}\,\not{\partial}\,\overline{\not{\partial}} &= -\overline{\not{\partial}}\,\overline{\not{\partial}}\,\not{\partial}-2\,\overline{\not{\partial}}\nabla^2=\nabla^2\,\not{\partial}-2\,\overline{\not{\partial}}\nabla^2=-\nabla^2\,\not{\partial}+2\nabla^2\gamma^0\partial_0\;, && (174)\\
\gamma_i\,\not{\partial}\gamma_i &= -\gamma_i\gamma_i\,\not{\partial}-2\,\overline{\not{\partial}}=(D-1)\,\not{\partial}-2\,\overline{\not{\partial}}=(D-3)\,\not{\partial}+2\gamma^0\partial_0\;, && (175)\\
\overline{\not{\partial}}\gamma_i\gamma_i &= -(D-1)\,\overline{\not{\partial}}=\gamma_i\gamma_i\,\overline{\not{\partial}}\;, && (176)\\
\gamma_i\,\overline{\not{\partial}}\gamma_i &= -\gamma_i\gamma_i\,\overline{\not{\partial}}-2\,\overline{\not{\partial}}=(D-3)\,\overline{\not{\partial}}\;. && (177)
\end{aligned}$$

Applying these identities gives,

$$\Big[\delta_{ij}\delta_{k\ell}+\delta_{ik}\delta_{j\ell}-\frac{2}{D-3}\delta_{i\ell}\delta_{jk}\Big]\times\Big\{D\partial_i\gamma_j\partial_k\,\not{\partial}\gamma_\ell-\partial^2\partial_i\gamma_j\gamma_k\gamma_\ell\Big\}$$
$$=\Big(D^2-\frac{2D}{D-3}\Big)\nabla^2\,\not{\partial}-4D\Big(\frac{D-4}{D-3}\Big)\nabla^2\,\overline{\not{\partial}}-\frac{4}{D-3}\partial^2\,\overline{\not{\partial}}\;. \quad (178)$$

For the second generic term the relevant contraction is,

$$\Big[\delta_{ij}\delta_{k\ell}+\delta_{ik}\delta_{j\ell}-\frac{2}{D-3}\delta_{i\ell}\delta_{jk}\Big]\times\partial_i\gamma_j\partial_k\,\not{\partial}\gamma_\ell$$
$$=\Big(\frac{D-5}{D-3}\Big)\,\overline{\not{\partial}}\,\not{\partial}\,\overline{\not{\partial}}+\nabla^2\gamma_i\,\not{\partial}\gamma_i\;, \quad (179)$$
$$=\Big(D-\frac{2}{D-3}\Big)\nabla^2\,\not{\partial}-4\Big(\frac{D-4}{D-3}\Big)\nabla^2\,\overline{\not{\partial}}\;. \quad (180)$$

In summing the contributions from Table 10 it is best to take advantage of cancellations between A_1 and A_2 terms. These occur between the 2nd and 3rd terms in the second column, the 4th and 5th terms of the 3rd column, and the 6th and 7th terms of the 3rd column. In each of these cases the result is finite; and it actually vanishes in the final case! Only the first term of column 2 and the 2nd term of column 3 contribute divergences. The result for the three contributions from [2–1] in Table 10 is,

$$-\frac{\kappa^2H^2\mu^{D-4}}{2^5\pi^{\frac{D}{2}}}\frac{(D-1)\Gamma(\frac{D}{2}+1)}{(D-3)^2(D-4)}\,(aa')^{2-\frac{D}{2}}\,\not{\partial}\,\delta^D(x-x')$$
$$+\frac{i\kappa^2H^2}{2^6\pi^4}\Big\{-\frac{3}{2}\partial^2\,\not{\partial}\Big[\frac{\ln(\mu^2\Delta x^2)}{\Delta x^2}\Big]+\partial^2\overline{\not{\partial}}\Big[\frac{4+2\ln(\frac{1}{4}H^2\Delta x^2)}{\Delta x^2}\Big]\Big\}+O(D-4). \quad (181)$$

Table 11: Residual $i\delta\Delta_A$ terms in which all derivatives act upon $\Delta x^2(x;x')$. All contributions are multiplied by $\frac{i\kappa^2 H^2}{2^6\pi^D}\Gamma(\frac{D}{2}{+}1)\Gamma(\frac{D}{2})(aa')^{2-\frac{D}{2}}$.

I	J	sub	$\frac{\gamma^\mu \Delta x_\mu}{\Delta x^{2D-2}}$	$\frac{\gamma^i \Delta x_i}{\Delta x^{2D-2}}$	$\frac{\Vert\Delta\vec{x}\Vert^2 \gamma^\mu \Delta x_\mu}{\Delta x^{2D}}$	$\frac{\Vert\Delta\vec{x}\Vert^2 \gamma^i \Delta x_i}{\Delta x^{2D}}$
2	3	a	$2(\frac{D-1}{D-3})$	$-\frac{2D}{D-3}$	0	0
2	3	b	0	1	$\frac{D^2}{2}$	$-2D$
2	3	c	0	$-(\frac{D-1}{D-3})$	$-\frac{D}{D-3}$	$\frac{2D}{D-3}$
3	1	a	$-\frac{4(D-1)(D-2)}{D-3}$	0	0	0
3	1	b	$2(\frac{D-1}{D-3})$	$2(\frac{D-4}{D-3})$	0	0
3	2	a	0	$4(\frac{D-2}{D-3})$	0	0
3	2	b	$-(\frac{D-1}{D-3})$	$(\frac{D+1}{D-3})$	$2(\frac{D-1}{D-3})$	$-4(\frac{D-1}{D-3})$
3	2	c	$\frac{1}{2}(D-1)$	$-\frac{1}{2}(D+1)$	$-(D-1)$	$2(D-1)$
3	2	d	$\frac{1}{2}(D-1)^2$	$-\frac{1}{2}(D+1)$	$-(D-1)^2$	$2(D-1)$
3	3	a	$2\frac{(D-1)(D-2)}{(D-3)}$	0	0	0
3	3	b	$-(\frac{D-1}{D-3})$	$-(\frac{D-4}{D-3})$	0	0
3	3	c	$-(\frac{D-1}{D-3})$	$-(\frac{D-4}{D-3})$	0	0
3	3	d	$-\frac{(D-1)(D-5)}{4(D-3)}$	$\frac{1}{2}(\frac{D-5}{D-3})$	$\frac{(D-5)(D-2)}{4(D-3)}$	$-\frac{(D-5)(D-2)}{2(D-3)}$
3	3	e	$-\frac{1}{4}(D-1)^2$	$\frac{1}{2}(D-1)$	$\frac{1}{4}(D-2)(D-1)$	$-\frac{1}{2}(D-2)$

The result for the five contributions from [2–2] in Table 10 is,

$$\frac{\kappa^2 H^2 \mu^{D-4}}{2^5 \pi^{\frac{D}{2}}} \frac{\Gamma(\frac{D}{2}{+}1)}{(D-3)^2(D-4)} (aa')^{2-\frac{D}{2}} \overline{\partial\!\!\!/}\, \delta^D(x-x') \\ + \frac{i\kappa^2 H^2}{2^6\pi^4}\Bigg\{\frac{1}{2}\partial^2 \overline{\partial\!\!\!/}\Big[\frac{\ln(\mu^2\Delta x^2)}{\Delta x^2}\Big] - \nabla^2 \partial\!\!\!/ \Big[\frac{2+\ln(\frac{1}{4}H^2\Delta x^2)}{\Delta x^2}\Big]\Bigg\} + O(D-4). \quad (182)$$

As might be expected from the similarities in their reductions, these two terms combine together nicely in the total for Table 10,

$$-i\Big[\Sigma^{T10}\Big](x;x') = \frac{\kappa^2 H^2 \mu^{D-4}}{2^5\pi^{\frac{D}{2}}} \frac{\Gamma(\frac{D}{2}{+}1)(aa')^{2-\frac{D}{2}}}{(D-3)^2(D-4)} \Big[-(D-1)\partial\!\!\!/ + \overline{\partial\!\!\!/}\Big]\delta^D(x-x') \\ + \frac{i\kappa^2 H^2}{2^6\pi^4}\Bigg\{\Big(-\frac{3}{2}\partial\!\!\!/\partial^2 + \frac{1}{2}\overline{\partial\!\!\!/}\,\partial^2\Big)\Big[\frac{\ln(\mu^2\Delta x^2)}{\Delta x^2}\Big]$$

$$+\Big(2\overline{\partial\!\!\!/}\,\partial^2 - \partial\!\!\!/\,\nabla^2\Big)\Big[\frac{2+\ln(\frac14 H^2\Delta x^2)}{\Delta x^2}\Big]\Bigg\} + O(D-4). \tag{183}$$

The next class is comprised of terms in which only the first series of $i\delta\Delta_A$ makes a nonzero contribution when all derivatives act upon the conformal coordinate separation. The results for this class of terms are summarized in Table 11. In reducing these terms the following derivatives occur many times,

$$\begin{aligned}
\partial_i i\delta\Delta_A(x;x') &= -\frac{H^2}{8\pi^{\frac{D}{2}}}\,\Gamma\Big(\frac{D}{2}+1\Big)(aa')^{2-\frac{D}{2}}\,\frac{\Delta x^i}{\Delta x^{D-2}} = -\partial'_i i\delta\Delta_A(x;x'), && (184)\\
\partial_0 i\delta\Delta_A(x;x') &= \frac{H^2}{8\pi^{\frac{D}{2}}}\,\Gamma\Big(\frac{D}{2}+1\Big)(aa')^{2-\frac{D}{2}}\Bigg\{\frac{\Delta\eta}{\Delta x^{D-2}} - \frac{aH}{2\Delta x^{D-4}}\Bigg\} \\
&\qquad + \frac{H^{D-2}}{2^D\pi^{\frac{D}{2}}}\,\frac{\Gamma(D-1)}{\Gamma(\frac{D}{2})}\,aH\ , && (185)\\
\partial'_0 i\delta\Delta_A(x;x') &= \frac{H^2}{8\pi^{\frac{D}{2}}}\,\Gamma\Big(\frac{D}{2}+1\Big)(aa')^{2-\frac{D}{2}}\Bigg\{-\frac{\Delta\eta}{\Delta x^{D-2}} - \frac{a'H}{2\Delta x^{D-4}}\Bigg\} \\
&\qquad + \frac{H^{D-2}}{2^D\pi^{\frac{D}{2}}}\,\frac{\Gamma(D-1)}{\Gamma(\frac{D}{2})}\,a'H\ . && (186)
\end{aligned}$$

We also make use of a number of gamma matrix identities,

$$\begin{aligned}
\gamma^\mu\gamma_\mu &= -D \quad \text{and} \quad \gamma^i\gamma^i = -(D-1)\ , && (187)\\
\gamma^\mu\gamma^\nu\gamma_\mu &= (D-2)\gamma^\nu \quad \text{and} \quad \gamma^i\gamma^\nu\gamma^i = (D-1)\gamma^\nu - 2\overline{\gamma}^\nu\ , && (188)\\
(\gamma^\mu\Delta x_\mu)^2 &= -\Delta x^2 \quad \text{and} \quad (\gamma^i\Delta x^i)^2 = -\|\Delta\vec{x}\|^2\ , && (189)\\
\gamma^i\gamma^\mu\Delta x_\mu\gamma^i &= (D-1)\gamma^\mu\Delta x_\mu - 2\gamma^i\Delta x^i\ , && (190)\\
\gamma^i\Delta x^i\gamma^\mu\Delta x_\mu\gamma^j\Delta x^j &= \|\Delta\vec{x}\|^2\gamma^\mu\Delta x_\mu - 2\|\Delta\vec{x}\|^2\gamma^i\Delta x^i\ . && (191)
\end{aligned}$$

In summing the many terms of Table 11 the constant $K \equiv D - 2/(D-3)$ occurs suspiciously often,

$$\begin{aligned}
-i\Big[\Sigma^{T11}\Big](x;x') &= \frac{i\kappa^2 H^2}{2^6\pi^D}\Gamma\Big(\frac{D}{2}+1\Big)\Gamma\Big(\frac{D}{2}\Big)(aa')^{2-\frac{D}{2}} \\
&\times\Bigg\{\Big[-2(D-1)+\Big(\frac{D-1}{4}\Big)K\Big]\frac{\gamma^\mu\Delta x_\mu}{\Delta x^{2D-2}} + \Big[-(D-2)+\frac{K}{2}\Big]\frac{\gamma^i\Delta x_i}{\Delta x^{2D-2}} \\
&\qquad -\Big(\frac{D-2}{4}\Big)K\frac{\|\Delta\vec{x}\|^2\gamma^\mu\Delta x_\mu}{\Delta x^{2D}} + \frac{(D-2)(D-4)}{(D-3)}\frac{\|\Delta\vec{x}\|^2\gamma^i\Delta x_i}{\Delta x^{2D}}\Bigg\}. && (192)
\end{aligned}$$

The last two terms can be reduced using the identities,

$$\frac{\|\Delta\vec{x}\|^2\gamma^\mu\Delta x_\mu}{\Delta x^{2D}} = \frac{1}{2}\frac{\gamma^\mu\Delta x_\mu}{\Delta x^{2D-2}} + \frac{1}{D-1}\frac{\gamma^i\Delta x_i}{\Delta x^{2D-2}} + \frac{\nabla^2}{4(D-2)(D-1)}\Big(\frac{\gamma^\mu\Delta x_\mu}{\Delta x^{2D-4}}\Big), \quad (193)$$

$$\frac{\|\Delta\vec{x}\|^2\gamma^i\Delta x_i}{\Delta x^{2D}} = \frac{1}{2}\Big(\frac{D+1}{D-1}\Big)\frac{\gamma^i\Delta x_i}{\Delta x^{2D-2}} + \frac{\nabla^2}{4(D-2)(D-1)}\Big(\frac{\gamma^i\Delta x_i}{\Delta x^{2D-4}}\Big). \quad (194)$$

Substituting these in Equation 192 gives,

$$-i\Big[\Sigma^{T11}\Big](x;x') = \frac{i\kappa^2H^2}{2^6\pi^D}\Gamma\Big(\frac{D}{2}+1\Big)\Gamma\Big(\frac{D}{2}\Big)(aa')^{2-\frac{D}{2}}\Bigg\{\Big[-2(D-1)+\frac{DK}{8}\Big] \times\frac{\gamma^\mu\Delta x_\mu}{\Delta x^{2D-2}} + \Big[-\frac{(D-2)(D^2-5D+10)}{2(D-1)(D-3)} + \frac{DK}{4(D-1)}\Big]\frac{\gamma^i\Delta x_i}{\Delta x^{2D-2}} - \frac{K\nabla^2}{16(D-1)}\Big(\frac{\gamma^\mu\Delta x_\mu}{\Delta x^{2D-4}}\Big) + \frac{(D-4)\nabla^2}{4(D-1)(D-3)}\frac{\gamma^i\Delta x_i}{\Delta x^{2D-4}}\Bigg\}. \quad (195)$$

We then apply the same formalism as in the previous sub-section to partially integrate, extract the local divergences and take $D=4$ for the remaining, integrable and ultraviolet finite nonlocal terms,

$$-i\Big[\Sigma^{T11}\Big](x;x') = \frac{\kappa^2H^2\mu^{D-4}}{2^7\pi^{\frac{D}{2}}}\frac{\Gamma(\frac{D}{2}+1)(aa')^{2-\frac{D}{2}}}{(D-3)(D-4)} \times\Bigg\{\Big[\frac{DK}{8}-2(D-1)\Big]\partial\!\!\!/+\Big[\frac{DK}{4(D-1)}-\frac{(D-2)(D^2-5D+10)}{2(D-1)(D-3)}\Big]\overline{\partial\!\!\!/}\Bigg\}\delta^D(x-x') + \frac{i\kappa^2H^2}{2^9\cdot 3\cdot\pi^4}\Bigg\{\Big[-15\partial\!\!\!/\,\partial^2-4\overline{\partial\!\!\!/}\,\partial^2\Big]\Big(\frac{\ln(\mu^2\Delta x^2)}{\Delta x^2}\Big)+\partial\!\!\!/\,\nabla^2\Big(\frac{1}{\Delta x^2}\Big)\Bigg\}+O(D-4). \quad (196)$$

The final class is comprised of terms in which one or more derivatives act upon a scale factor. Within this class we report contributions from the first series in Table 12 and contributions from the second series in Table 13. Each nonzero entry in the 4th and 5th columns of Table 12 diverges logarithmically like $1/\Delta x^{2D-4}$. However, the sum in each case results in an additional factor of $a-a'=aa'H\Delta\eta$ which makes the contribution from Table 12 integrable,

$$-i\Big[\Sigma^{T12}\Big](x;x') = \frac{i\kappa^2H^4}{2^6\pi^D}\Gamma\Big(\frac{D}{2}+1\Big)\Gamma\Big(\frac{D}{2}\Big)(aa')^{3-\frac{D}{2}}\Bigg\{-\Big(\frac{D-1}{D-3}\Big)\frac{\gamma^0\Delta\eta}{\Delta x^{2D-4}} -\frac{1}{2}\Big(\frac{3D-4}{D-3}\Big)\frac{\gamma^i\Delta x_i\gamma^\mu\Delta x_\mu\gamma^0\Delta\eta}{\Delta x^{2D-2}} + \frac{(D-1)(D-4)}{4(D-3)}\frac{\gamma^\mu\Delta x_\mu}{\Delta x^{2D-4}}\Bigg\}. \quad (197)$$

Table 12: Residual $i\delta\Delta_A$ terms in which some derivatives act upon the scale factors of the first series. The factor $\frac{i\kappa^2 H^2}{2^6 \pi^D}\Gamma(\frac{D}{2}{+}1)\Gamma(\frac{D}{2})(aa')^{2-\frac{D}{2}}$ multiplies all contributions.

I	J	sub	$\frac{H\gamma^0}{\Delta x^{2D-4}}$	$\frac{H\gamma^i \Delta x_i \gamma^\mu \Delta x_\mu \gamma^0}{\Delta x^{2D-2}}$	$\frac{H^2 a a' \gamma^\mu \Delta x_\mu}{\Delta x^{2D-4}}$
2	1		$2(\frac{D-1}{D-3})a'$	$(\frac{2D}{D-3})a'$	0
2	3	a	$-(\frac{D-1}{D-3})a'$	$(\frac{-D}{D-3})a'$	0
3	1	a	0	0	$\frac{(D-1)(D-4)}{2(D-3)}$
3	1	b	0	$(\frac{D-4}{D-3})a'$	0
3	2	a	$-(\frac{D-1}{D-3})a$	$-2(\frac{D-2}{D-3})a$	0
3	3	a	0	0	$-\frac{(D-1)(D-4)}{4(D-3)}$
3	3	b	0	$\frac{1}{2}(\frac{D-4}{D-3})a$	0
3	3	c	0	$-\frac{1}{2}(\frac{D-4}{D-3})a'$	0

Table 13: Residual $i\delta\Delta_A$ terms in which some derivatives act upon the scale factors of the second series. All contributions are multiplied by $\frac{i\kappa^2 H^{D-2}}{2^{D+2}\pi^D}\Gamma(D{-}1)$.

I	J	sub	$\frac{H\gamma^0}{\Delta x^D}$	$\frac{H\gamma^i \Delta x_i \gamma^\mu \Delta x_\mu \gamma^0}{\Delta x^{D+2}}$	$\partial^2(\frac{H\gamma^0}{\Delta x^{D-2}})$
2	1		$-2(\frac{D-1}{D-3})a'$	$-(\frac{2D}{D-3})a'$	0
2	3	a	$(\frac{D-1}{D-3})a'$	$(\frac{D}{D-3})a'$	0
3	1	a	0	0	$\frac{(D-1)\,a}{(D-2)(D-3)}$
3	2	a	$(\frac{D-1}{D-3})a$	$(\frac{D}{D-3})a$	0

This is another example of the fact that the self-energy is odd under interchange of x^μ and x'^μ.

The same thing happens with the contribution from Table 13,

$$-i\Big[\Sigma^{T13}\Big](x;x') = \frac{i\kappa^2 H^D}{2^{D+2}\pi^D}\Gamma(D-1)aa'\Bigg\{\Big(\frac{D-1}{D-3}\Big)\frac{\gamma^0\Delta\eta}{\Delta x^D} + \Big(\frac{D}{D-3}\Big)\frac{\gamma^i \Delta x_i \gamma^\mu \Delta x_\mu \gamma^0 \Delta\eta}{\Delta x^{D+2}} + \gamma^0\Big(\frac{D-1}{D-3}\Big)\frac{i2\pi^{\frac{D}{2}}}{\Gamma(\frac{D}{2})}\frac{\delta^D(x-x')}{Ha}\Bigg\}. \quad (198)$$

We can therefore set $D\!=\!4$, at which point the two Tables cancel except for the delta function term,

$$-i\Big[\Sigma^T 12{+}13\Big](x;x') = \frac{\kappa^2 H^{D-2}}{(4\pi)^{\frac{D}{2}}} \frac{\Gamma(D\!-\!1)}{\Gamma(\frac{D}{2})} \times -\frac{1}{2}\Big(\frac{D\!-\!1}{D\!-\!3}\Big) a H \gamma^0 \delta^D(x\!-\!x') \!+\! O(D\!-\!4). \tag{199}$$

It is worth commenting that this term violates the reflection symmetry of Equation 103. In $D\!=\!4$ it cancels the similar term in Equation 127.

3.5 Sub-Leading Contributions from $i\delta\Delta_B$

In this subsection we work out the contribution from substituting the residual B-type part of the graviton propagator in Table 5,

$$i\Big[{}_{\alpha\beta}\Delta_{\rho\sigma}\Big] \longrightarrow -\Big[\delta^0_\alpha \delta^0_\sigma \overline{\eta}_{\beta\rho} + \delta^0_\alpha \delta^0_\rho \overline{\eta}_{\beta\sigma} + \delta^0_\beta \delta^0_\sigma \overline{\eta}_{\alpha\rho} + \delta^0_\beta \delta^0_\rho \overline{\eta}_{\alpha\sigma}\Big] i\delta\Delta_B \; . \tag{200}$$

As in the two previous sub-sections we first make the requisite contractions and then act the derivatives. The result of this first step is summarized in Table 14. We have sometimes broken the result for a single vertex pair into parts because the four different tensors in (200) can make distinct contributions, and because distinct contributions also come from breaking up factors of $\gamma^\alpha J^{\beta\mu}$. These distinct contributions are labeled by subscripts a, b, c, etc.

$i\delta\Delta_B(x;x')$ is the residual of the B-type propagator of Equation 84 after the conformal contribution has been subtracted,

$$\begin{aligned} i\delta\Delta_B(x;x') = {} & \frac{H^2 \Gamma(\frac{D}{2})}{16\pi^{\frac{D}{2}}} \frac{(aa')^{2-\frac{D}{2}}}{\Delta x^{D-4}} - \frac{H^{D-2}}{(4\pi)^{\frac{D}{2}}} \frac{\Gamma(D\!-\!2)}{\Gamma\Big(\frac{D}{2}\Big)} \\ & + \frac{H^{D-2}}{(4\pi)^{\frac{D}{2}}} \sum_{n=1}^{\infty} \Bigg\{ \frac{\Gamma(n\!+\!\frac{D}{2})}{\Gamma(n\!+\!2)} \Big(\frac{y}{4}\Big)^{n-\frac{D}{2}+2} - \frac{\Gamma(n\!+\!D\!-\!2)}{\Gamma(n\!+\!\frac{D}{2})} \Big(\frac{y}{4}\Big)^n \Bigg\} . \end{aligned} \tag{201}$$

As was the case for the $i\delta\Delta_A(x;x')$ contributions considered in the previous sub-section, this diagram is not sufficiently singular for the infinite series terms from $i\delta\Delta_B(x;x')$ to make a nonzero contribution in the $D=4$ limit. Unlike $i\delta\Delta_A(x;x')$, even the $n\!=\!0$ terms of $i\delta\Delta_B(x;x')$ vanish for $D\!=\!4$. This means they can only contribute when multiplied by a divergence.

Contributions from the [2–2] vertex pair require special treatment to take advantage of the cancelation between the two series. We will work out the

Table 14: Contractions from the $i\delta\Delta_B$ part of the graviton propagator.

I	J	sub	$iV_I^{\alpha\beta}(x)\, i[S](x;x')\, iV_J^{\rho\sigma}(x')\, [{}_{\alpha\beta}T^B_{\rho\sigma}]\, i\delta\Delta_B(x;x')$
2	1		0
2	2	a	$-\frac{1}{2}\kappa^2\partial_0'\{\gamma^{(0}\partial^{k)}i[S](x;x')\gamma_k i\delta\Delta_B(x;x')\}$
2	2	b	$-\frac{1}{2}\kappa^2\partial_k\{\gamma^{(0}\partial^{k)}i[S](x;x')\gamma^0 i\delta\Delta_B(x;x')\}$
2	3	a	$-\frac{1}{8}\kappa^2\gamma_k\partial_0 i[S](x;x')\gamma^k\, \partial_0' i\delta\Delta_B(x;x')$
2	3	b	$\frac{1}{8}\kappa^2\gamma^0\partial_0' i\delta\Delta_B(x;x')\, \partial_k i[S](x;x')\gamma^k$
2	3	c	$-\frac{1}{8}\kappa^2\gamma^k\partial_k i\delta\Delta_B(x;x')\, \partial_0 i[S](x;x')\gamma^0$
2	3	d	$\frac{1}{8}\kappa^2\gamma^0\partial^k i[S](x;x')\gamma^0\, \partial_k i\delta\Delta_B(x;x')$
3	1		0
3	2	a	$\frac{1}{8}\kappa^2\partial_0'\{\gamma^k i[S](x;x')\gamma_k\, \partial_0 i\delta\Delta_B(x;x')\}$
3	2	b	$\frac{1}{8}\kappa^2\gamma^k\partial_k\{i[S](x;x')\gamma^0\, \partial_0 i\delta\Delta_B(x;x')\}$
3	2	c	$-\frac{1}{8}\kappa^2\gamma^0\partial_0'\{i[S](x;x')\gamma^k\, \partial_k i\delta\Delta_B(x;x')\}$
3	2	d	$-\frac{1}{8}\kappa^2\partial_k\{\gamma^0 i[S](x;x')\gamma^0\, \partial^k i\delta\Delta_B(x;x')\}$
3	3	a	$-\frac{1}{16}\kappa^2\gamma_k i[S](x;x')\gamma^k\partial_0\partial_0' i\delta\Delta_B(x;x')$
3	3	b	$\frac{1}{16}\kappa^2\gamma^0 i[S](x;x')\gamma^k\, \partial_k\partial_0' i\delta\Delta_B(x;x')$
3	3	c	$-\frac{1}{16}\kappa^2\gamma^k i[S](x;x')\gamma^0\, \partial_0\partial_k i\delta\Delta_B(x;x')$
3	3	d	$\frac{1}{16}\kappa^2\gamma^0 i[S](x;x')\gamma^0\nabla^2 i\delta\Delta_B(x;x')$

"a" term from Table 14,

$$\Big[2\text{–}2\Big]_a = -\frac{i\kappa^2\Gamma(\frac{D}{2}-1)}{16\pi^{\frac{D}{2}}}\partial_0'\Big\{i\delta\Delta_B(x;x')\Big(\gamma^0\not{\partial}\overline{\not{\partial}}-\gamma^i\not{\partial}\gamma^i\partial_0\Big)\Big[\frac{1}{\Delta x^{D-2}}\Big]\Big\}, \qquad (202)$$

$$= \frac{i\kappa^2\Gamma(\frac{D}{2}-1)}{16\pi^{\frac{D}{2}}}\partial_0'\Big\{i\delta\Delta_B(x;x')\Big(-3\partial_0\overline{\not{\partial}}+\gamma^0\nabla^2+(D-1)\not{\partial}\partial_0\Big)\Big[\frac{1}{\Delta x^{D-2}}\Big]\Big\}. \qquad (203)$$

A key identity for reducing the $[2-2]$ terms involves commuting two derivatives through $1/\Delta x^{D-4}$,

$$\frac{1}{\Delta x^{D-4}}\partial_\mu\partial_\nu\Big[\frac{1}{\Delta x^{D-2}}\Big] = \frac{1}{4(D-3)}\Big(-\eta_{\mu\nu}\partial^2+D\partial_\mu\partial_\nu\Big)\Big[\frac{1}{\Delta x^{2D-6}}\Big]\ . \qquad (204)$$

This can be used to extract the derivatives from the first term of $i\delta\Delta_B(x;x')$,

Table 15: Residual $i\delta\Delta_B$ terms in which all derivatives act upon $\Delta x^2(x;x')$. All contributions are multiplied by $\frac{i\kappa^2 H^2}{2^8\pi^D}\Gamma^2(\frac{D}{2})(D-4)(aa')^{2-\frac{D}{2}}$.

I	J	sub	$\frac{\gamma^\mu \Delta x_\mu}{\Delta x^{2D-2}}$	$\frac{\gamma^i \Delta x_i}{\Delta x^{2D-2}}$	$\frac{\Vert\Delta\vec{x}\Vert^2\gamma^\mu\Delta x_\mu}{\Delta x^{2D}}$	$\frac{\Vert\Delta\vec{x}\Vert^2\gamma^i\Delta x_i}{\Delta x^{2D}}$
2	3	a	$(D-1)^2$	$-(D+1)$	$-D(D-1)$	$2D$
2	3	b	$(D-1)$	$-2D+1$	$-D$	$2D$
2	3	c	0	$-(D-1)$	$-D$	$2D$
2	3	d	0	-1	$-D$	$2D$
3	2	a	$-2(D-1)(D-2)$	$3D-5$	$2(D-1)^2$	$-4(D-1)$
3	2	b	$-(D-1)$	$3(D-1)$	$2(D-1)$	$-4(D-1)$
3	2	c	0	$2D-3$	$2(D-1)$	$-4(D-1)$
3	2	d	$-(D-1)$	$2D-1$	$2(D-1)$	$-4(D-1)$
3	3	a	$\frac{1}{2}(D-1)(D-3)$	$-(D-3)$	$-\frac{1}{2}(D-1)(D-2)$	$(D-2)$
3	3	b	0	$-\frac{1}{2}(D-2)$	$-\frac{1}{2}(D-2)$	$(D-2)$
3	3	c	0	$-\frac{1}{2}(D-2)$	$-\frac{1}{2}(D-2)$	$(D-2)$
3	3	d	$\frac{1}{2}(D-1)$	$-(D-1)$	$-\frac{1}{2}(D-2)$	$(D-2)$
Total			$-\frac{1}{2}(D-1)(D-2)$	$3(D-2)$	$\frac{1}{2}(D+2)(D-2)$	$-4(D-2)$

at which point the result is integrable and we can take $D\!=\!4$,

$$\Big[2\text{–}2\Big]^1_a = \frac{i\kappa^2 H^2}{2^8\pi^D}\Gamma\Big(\frac{D}{2}\Big)\Gamma\Big(\frac{D}{2}-1\Big) \times\partial_0'\Bigg\{\frac{(aa')^{2-\frac{D}{2}}}{\Delta x^{D-4}}\Big(-3\partial_0\overline{\partial}\!\!\!/+\gamma^0\nabla^2+(D-1)\partial\!\!\!/\partial_0\Big)\Big[\frac{1}{\Delta x^{D-2}}\Big]\Bigg\}, \tag{205}$$

$$= \frac{i\kappa^2 H^2}{2^9\pi^D}\frac{\Gamma(\frac{D}{2}+1)\Gamma(\frac{D}{2}-1)}{D-3}(aa')^{2-\frac{D}{2}} \times\Big(-\partial_0-\frac{1}{2}(D-4)Ha'\Big)\Big(-3\partial_0\overline{\partial}\!\!\!/+\gamma^0\nabla^2+(D-1)\partial\!\!\!/\partial_0\Big)\Big[\frac{1}{\Delta x^{2D-6}}\Big], \tag{206}$$

$$= -\frac{i\kappa^2 H^2}{2^8\pi^4}\gamma^0\partial_0\Big(3\partial_0^2+\nabla^2\Big)\Big[\frac{1}{\Delta x^2}\Big] + O(D-4)\;. \tag{207}$$

Of course the second term of $i\delta\Delta_B$ is constant so the derivatives are already

extracted,

$$\Big[2\text{–}2\Big]^2_a = \frac{i\kappa^2 H^{D-2}}{2^{D+3}\pi^D}\frac{\Gamma(D-2)}{D-2}\partial_0\Big(-3\partial_0\overline{\partial\!\!\!/}+\gamma^0\nabla^2+(D-1)\partial\!\!\!/\partial_0\Big)\Big[\frac{1}{\Delta x^{D-2}}\Big], \tag{208}$$

$$= \frac{i\kappa^2 H^2}{2^8\pi^4}\gamma^0\partial_0\Big(3\partial_0^2+\nabla^2\Big)\Big[\frac{1}{\Delta x^2}\Big] + O(D-4) \; . \tag{209}$$

Hence the total for $[2\text{–}2]_a$ is zero in $D\!=\!4$ dimensions!

The analogous result for the initial reduction of the other [2–2] term is,

$$\Big[2\text{–}2\Big]_b = \frac{i\kappa^2\Gamma(\frac{D}{2}-1)}{16\pi^{\frac{D}{2}}} \times\partial_k\Bigg\{i\delta\Delta_B(x;x')\Big(-\gamma^0\partial_0\partial_k+\overline{\partial\!\!\!/}\,\partial_k+\gamma_k\partial_0^2+\gamma_k\overline{\partial\!\!\!/}\gamma^0\partial_0\Big)\Big[\frac{1}{\Delta x^{D-2}}\Big]\Bigg\}. \tag{210}$$

The results for each of the two terms of $i\delta\Delta_B$ are,

$$\Big[2\text{–}2\Big]^1_b = \frac{i\kappa^2 H^2}{2^9\pi^D}\frac{\Gamma(\frac{D}{2}+1)\Gamma(\frac{D}{2}-1)}{D-3}(aa')^{2-\frac{D}{2}} \times\Big(-2\gamma^0\partial_0\nabla^2+\overline{\partial\!\!\!/}\,\nabla^2+\overline{\partial\!\!\!/}\,\partial_0^2\Big)\Big[\frac{1}{\Delta x^{2D-6}}\Big], \tag{211}$$

$$= \frac{i\kappa^2 H^2}{2^8\pi^4}\Big(-2\gamma^0\partial_0\nabla^2+\overline{\partial\!\!\!/}\,\nabla^2+\overline{\partial\!\!\!/}\,\partial_0^2\Big)\Big[\frac{1}{\Delta x^2}\Big] + O(D-4) \; , \tag{212}$$

$$\Big[2\text{–}2\Big]^2_b = \frac{i\kappa^2 H^{D-2}}{2^{D+3}\pi^D}\frac{\Gamma(D-2)}{D-2}\Big(2\gamma^0\partial_0\nabla^2-\overline{\partial\!\!\!/}\,\nabla^2-\overline{\partial\!\!\!/}\,\partial_0^2\Big)\Big[\frac{1}{\Delta x^{D-2}}\Big], \tag{213}$$

$$= \frac{i\kappa^2 H^2}{2^8\pi^4}\Big(2\gamma^0\partial_0\nabla^2-\overline{\partial\!\!\!/}\,\nabla^2-\overline{\partial\!\!\!/}\,\partial_0^2\Big)\Big[\frac{1}{\Delta x^2}\Big] + O(D-4) \; . \tag{214}$$

Hence the entire contribution from [2–2] vanishes in $D\!=\!4$.

The lower vertex pairs all involve at least one derivative of $i\delta\Delta_B$,

$$\partial_i i\delta\Delta_B(x;x') = -\frac{H^2\Gamma(\frac{D}{2})}{16\pi^{\frac{D}{2}}}(D-4)(aa')^{2-\frac{D}{2}}\,\frac{\Delta x^i}{\Delta x^{D-2}} = -\partial_i' \delta\Delta_B(x;x') \; , \tag{215}$$

$$\partial_0 i\delta\Delta_B(x;x') = \frac{H^2\Gamma(\frac{D}{2})}{16\pi^{\frac{D}{2}}}(D-4)(aa')^{2-\frac{D}{2}}\Bigg\{\frac{\Delta\eta}{\Delta x^{D-2}}-\frac{aH}{2\Delta x^{D-4}}\Bigg\}, \tag{216}$$

$$\partial_0' i\delta\Delta_B(x;x') = \frac{H^2\Gamma(\frac{D}{2})}{16\pi^{\frac{D}{2}}}(D-4)(aa')^{2-\frac{D}{2}}\Bigg\{-\frac{\Delta\eta}{\Delta x^{D-2}}-\frac{a'H}{2\Delta x^{D-4}}\Bigg\}. \tag{217}$$

These reductions are very similar to those of the analogous $i\delta\Delta_A$ terms. We make use of the same gamma matrix identities of Equations 187-191 that

were used in the previous sub-section. The only really new feature is that one sometimes encounters factors of $\Delta\eta^2$ which we always resolve as,

$$\Delta\eta^2 = -\Delta x^2 + \|\Delta\vec{x}\|^2 \; . \tag{218}$$

Table 15 gives our results for the most singular contributions, those in which all derivatives act upon the conformal coordinate separation Δx^2.

The only really unexpected thing about Table 15 is the overall factor of $(D-2)$ common to each of the four sums,

$$-i\Big[\Sigma^{T15}\Big](x;x') = \frac{i\kappa^2 H^2}{2^8\pi^D}\Gamma^2\Big(\frac{D}{2}\Big)(D-2)(D-4)(aa')^{2-\frac{D}{2}}\Bigg\{-\frac{1}{2}(D-1)\frac{\gamma^\mu\Delta x_\mu}{\Delta x^{2D-2}} +3\frac{\gamma^i\Delta x_i}{\Delta x^{2D-2}} + \frac{1}{2}(D+2)\frac{\|\Delta\vec{x}\|^2\gamma^\mu\Delta x_\mu}{\Delta x^{2D}} - 4\frac{\|\Delta\vec{x}\|^2\gamma^i\Delta x_i}{\Delta x^{2D}}\Bigg\}. \tag{219}$$

As with the result of Table 11, we use the differential identities 193-194 to prepare the last two terms for partial integration,

$$\begin{aligned}
-i\Big[\Sigma^{T15}\Big](x;x') &= \frac{i\kappa^2 H^2}{2^8\pi^D}\Gamma^2\Big(\frac{D}{2}\Big)(D-2)(D-4)(aa')^{2-\frac{D}{2}} \\
&\times\Bigg\{-\frac{1}{4}(D-4)\frac{\gamma^\mu\Delta x_\mu}{\Delta x^{2D-2}} + \frac{1}{2}\Big(\frac{3D-8}{D-1}\Big)\frac{\gamma^i\Delta x_i}{\Delta x^{2D-2}} \\
&\qquad + \frac{(D+2)\,\nabla^2}{8(D-1)(D-2)}\Big(\frac{\gamma^\mu\Delta x_\mu}{\Delta x^{2D-4}}\Big) - \frac{\nabla^2}{(D-1)(D-2)}\Big(\frac{\gamma^i\Delta x_i}{\Delta x^{2D-4}}\Big)\Bigg\}, \\
&= \frac{i\kappa^2 H^2}{2^8\pi^D}\Gamma^2\Big(\frac{D}{2}\Big)(aa')^{2-\frac{D}{2}}\Bigg\{\frac{1}{16}\Big(\frac{D-4}{D-3}\Big)\not{\partial}\partial^2 - \frac{1}{8}\frac{(3D-8)}{(D-1)(D-3)}\overline{\not{\partial}}\,\partial^2 \\
&\qquad -\frac{1}{16}\frac{(D+2)(D-4)}{(D-1)(D-3)}\not{\partial}\nabla^2 + \frac{1}{2}\frac{(D-4)}{(D-1)(D-3)}\overline{\not{\partial}}\,\nabla^2\Bigg\}\frac{1}{\Delta x^{2D-6}}.
\end{aligned} \tag{220}$$

The expression is now integrable so we can take $D=4$,

$$-\,i\Big[\Sigma^{T15}\Big](x;x') = \frac{i\kappa^2 H^2}{2^8\pi^4}\Big\{-\frac{1}{6}\overline{\not{\partial}}\,\partial^2\Big\}\frac{1}{\Delta x^2} + O(D-4) \; . \tag{221}$$

Unlike the $i\delta\Delta_A$ terms there is no net contribution when one or more of the derivatives acts upon a scale factor. If both derivatives act on scale factors the result is integrable in $D=4$ dimensions, and vanishes owing to the factor of $(D-4)^2$ from differentiating both $a^{2-\frac{D}{2}}$ and $a'^{2-\frac{D}{2}}$. If a single derivative acts upon a scale factor, the result is a factor of either $(D-4)a$

or $(D-4)a'$ times a term which is logarithmically divergent and *even* under interchange of x^μ and x'^μ. As we have by now seen many times, the sum of all such terms contrives to obey reflection symmetry of Equation 103 by the separate extra factors of $(D-4)a$ and $(D-4)a'$ combining to give,

$$(D-4)(a-a') = (D-4)aa'H\Delta\eta \; . \tag{222}$$

Of course this makes the sum integrable in $D=4$ dimensions, at which point we can take $D=4$ and the result vanishes on account of the overall factor of $(D-4)$.

3.6 Sub-Leading Contributions from $i\delta\Delta_C$

The point of this subsection is to compute the contribution from replacing the graviton propagator in Table 5 by its residual C-type part,

$$i\Big[{}_{\alpha\beta}\Delta_{\rho\sigma}\Big] \rightarrow 2\Bigg[\frac{\eta_{\alpha\beta}\eta_{\rho\sigma}}{(D-2)(D-3)} + \frac{\delta^0_\alpha\delta^0_\beta\eta_{\rho\sigma} + \eta_{\alpha\beta}\delta^0_\rho\delta^0_\sigma}{D-3} + \Big(\frac{D-2}{D-3}\Big)\delta^0_\alpha\delta^0_\beta\delta^0_\rho\delta^0_\sigma\Bigg] i\delta\Delta_C. \tag{223}$$

As in the previous sub-sections we first make the requisite contractions and then act the derivatives. The result of this first step is summarized in Table 16. We have sometimes broken the result for a single vertex pair into parts because the four different tensors in Equation 223 can make distinct contributions, and because distinct contributions also come from breaking up factors of $\gamma^\alpha J^{\beta\mu}$. These distinct contributions are labeled by subscripts a, b, c, etc.

Here $i\delta\Delta_C(x;x')$ is the residual of the C-type propagator of Equation 85 after the conformal contribution has been subtracted,

$$i\delta\Delta_C(x;x') = \frac{H^2}{16\pi^{\frac{D}{2}}}\Big(\frac{D}{2}-3\Big)\Gamma\Big(\frac{D}{2}-1\Big)\frac{(aa')^{2-\frac{D}{2}}}{\Delta x^{D-4}} + \frac{H^{D-2}}{(4\pi)^{\frac{D}{2}}}\frac{\Gamma(D-3)}{\Gamma(\frac{D}{2})}$$
$$-\frac{H^{D-2}}{(4\pi)^{\frac{D}{2}}}\sum_{n=1}^{\infty}\Bigg\{\Big(n-\frac{D}{2}+3\Big)\frac{\Gamma(n+\frac{D}{2}-1)}{\Gamma(n+2)}\Big(\frac{y}{4}\Big)^{n-\frac{D}{2}+2} - (n+1)\frac{\Gamma(n+D-3)}{\Gamma(n+\frac{D}{2})}\Big(\frac{y}{4}\Big)^{n}\Bigg\}. \tag{224}$$

As with the contributions from $i\delta\Delta_B(x;x')$ considered in the previous subsection, the only way $i\delta\Delta_C(x;x')$ can give a nonzero contribution in $D=4$ dimensions is for it to multiply a singular term. That means only the $n=0$ term can possibly contribute. Even for the $n=0$ term, both derivatives must act upon a Δx^2 to make a nonzero contribution in $D=4$ dimensions.

Table 16: Contractions from the $i\delta\Delta_C$ part of the graviton propagator.

I	J	sub	$iV_I^{\alpha\beta}(x)\, i[S](x;x')\, iV_J^{\rho\sigma}(x')\, [_{\alpha\beta}T^C_{\rho\sigma}]\, i\delta\Delta_C(x;x')$
2	1	a	$-\frac{1}{(D-3)(D-2)}\kappa^2 \not{\partial}\, \delta^D(x-x')\, i\delta\Delta_C(x;x)$
2	1	b	$-\frac{1}{D-3}\kappa^2 \partial'_\mu \{\gamma^0 \partial_0\, i[S](x;x')\gamma^\mu\, i\delta\Delta_C(x;x')\}$
2	2	a	$\frac{1}{2(D-3)(D-2)}\kappa^2 \not{\partial}\, \delta^D(x-x')\, i\delta\Delta_C(x;x)$
2	2	b	$-\frac{1}{2(D-3)}\kappa^2 \gamma^0 \partial_0\, \delta^D(x-x')\, i\delta\Delta_C(x;x)$
2	2	c	$+\frac{1}{2(D-3)}\kappa^2 \partial'_\mu \{\gamma^0 \partial_0\, i[S](x;x')\gamma^\mu\, i\delta\Delta_C(x;x')\}$
2	2	d	$-\frac{1}{2}(\frac{D-2}{D-3})\kappa^2 \partial'_0 \{\gamma^0 \partial_0\, i[S](x;x')\gamma^0\, i\delta\Delta_C(x;x')\}$
2	3	a	$-\frac{(D-1)}{4(D-3)(D-2)}\kappa^2\, \delta^D(x-x')\, \not{\partial}'\, i\delta\Delta_C(x;x')$
2	3	b	$+\frac{1}{4(D-3)}\kappa^2\, \delta^D(x-x')\gamma^i \partial'_i\, i\delta\Delta_C(x;x')$
2	3	c	$+\frac{1}{4}(\frac{D-1}{D-3})\kappa^2 \gamma^0 \partial_0\, i[S](x;x')\, \not{\partial}'\, i\delta\Delta_C(x;x')$
2	3	d	$-\frac{1}{4}(\frac{D-2}{D-3})\kappa^2 \gamma^0 \partial_0\, i[S](x;x')\gamma^i \partial'_i\, i\delta\Delta_C(x;x')$
3	1	a	$-\frac{(D-1)}{2(D-3)(D-2)}\kappa^2 \partial'_\mu \{\not{\partial}\, i\delta\Delta_C(x;x')\, i[S](x;x')\gamma^\mu\}$
3	1	b	$+\frac{1}{2(D-3)}\kappa^2 \partial'_\mu \{\gamma^i \partial_i\, i\delta\Delta_C(x;x')\, i[S](x;x')\gamma^\mu\}$
3	2	a	$\frac{(D-1)}{4(D-3)(D-2)}\kappa^2 \partial'_\mu \{\not{\partial}\, i\delta\Delta_C(x;x')\, i[S](x;x')\gamma^\mu\}$
3	2	b	$-\frac{1}{4}(\frac{D-1}{D-3})\kappa^2 \partial'_0 \{\not{\partial}\, i\delta\Delta_C(x;x')\, i[S](x;x')\gamma^0\}$
3	2	c	$-\frac{1}{4(D-3)}\kappa^2 \partial'_\mu \{\gamma^i \partial_i\, i\delta\Delta_C(x;x')\, i[S](x;x')\gamma^\mu\}$
3	2	d	$+\frac{1}{4}(\frac{D-2}{D-3})\kappa^2 \partial'_0 \{\gamma^i \partial_i\, i\delta\Delta_C(x;x')\, i[S](x;x')\gamma^0\}$
3	3	a	$\frac{(D-1)^2}{8(D-3)(D-2)}\kappa^2 \gamma^\mu i[S](x;x')\partial_\mu \not{\partial}'\, i\delta\Delta_C(x;x')$
3	3	b	$-\frac{1}{8}(\frac{D-1}{D-3})\kappa^2 \gamma^\mu i[S](x;x')\partial_\mu \gamma^j \partial'_j\, i\delta\Delta_C(x;x')$
3	3	c	$-\frac{1}{8}(\frac{D-1}{D-3})\kappa^2 \gamma^i i[S](x;x')\partial_i\, \not{\partial}'\, i\delta\Delta_C(x;x')$
3	3	d	$+\frac{1}{8}(\frac{D-2}{D-3})\kappa^2 \gamma^i i[S](x;x')\partial_i \gamma^j \partial'_j\, i\delta\Delta_C(x;x')$

Those of the [2–1] and [2–2] vertex pairs which are not proportional to delta functions after the initial contraction of Table 16 all contrive to give delta functions in the end. This happens through the same key identity 204 which was used to reduce the analogous terms in the previous subsection. In each case we have finite constants times different contractions of the following tensor function,

$$\partial'_\mu \Bigl\{ i\delta\Delta_C(x;x') \partial_\alpha \partial_\beta \Bigl[\frac{1}{\Delta x^{D-2}} \Bigr] \Bigr\} = \frac{H^{D-2}}{(4\pi)^{\frac{D}{2}}} \frac{\Gamma(D-3)}{\Gamma(\frac{D}{2})} \, \partial'_\mu \partial_\alpha \partial_\beta \Bigl[\frac{1}{\Delta x^{D-2}} \Bigr]$$

$$+ \frac{H^2}{16\pi^{\frac{D}{2}}} \Bigl(\frac{D}{2} - 3 \Bigr) \Gamma\Bigl(\frac{D}{2} - 1 \Bigr) \partial'_\mu \Bigl\{ \frac{(aa')^{2-\frac{D}{2}}}{\Delta x^{D-4}} \, \partial_\alpha \partial_\beta \Bigl[\frac{1}{\Delta x^{D-2}} \Bigr] \Bigr\}, \quad (225)$$

$$= \frac{H^{D-2}}{(4\pi)^{\frac{D}{2}}} \frac{\Gamma(D-3)}{\Gamma(\frac{D}{2})} \, \partial'_\mu \partial_\alpha \partial_\beta \Bigl[\frac{1}{\Delta x^{D-2}} \Bigr] + \frac{H^{D-2}}{16\pi^{\frac{D}{2}}} \Bigl(\frac{D}{2} - 3 \Bigr) \Gamma\Bigl(\frac{D}{2} - 1 \Bigr) (aa')^{2-\frac{D}{2}}$$

$$\times \Bigl(\partial'_\mu - \frac{1}{2}(D-4) H a' \Bigr) \Bigl\{ \frac{D \, \partial_\alpha \partial_\beta}{4(D-3)} - \frac{\eta_{\alpha\beta} \partial^2}{4(D-3)} \Bigr\} \Bigl[\frac{1}{\Delta x^{2D-6}} \Bigr], \quad (226)$$

$$= \frac{H^2}{16\pi^2} \partial'_\mu \partial_\alpha \partial_\beta \Bigl[\frac{1}{\Delta x^2} \Bigr] - \frac{H^2}{16\pi^2} \partial'_\mu \Bigl(\partial_\alpha \partial_\beta - \frac{1}{4} \eta_{\alpha\beta} \partial^2 \Bigr) \Bigl[\frac{1}{\Delta x^2} \Bigr] + O(D-4), \quad (227)$$

$$= -\frac{iH^2}{16} \, \eta_{\alpha\beta} \partial_\mu \delta^4(x-x') + O(D-4) \; . \quad (228)$$

It remains to multiply Equation 228 by the appropriate prefactors and take the appropriate contraction. For example, the $[2\text{–}1]_b$ contribution is,

$$-\frac{\kappa^2}{D-3} \times \frac{i\Gamma(\frac{D}{2}-1)}{4\pi^{\frac{D}{2}}} \times \gamma^0 \delta^\alpha_0 \gamma^\beta \gamma^\mu \times -\frac{iH^2}{16} \, \eta_{\alpha\beta} \partial_\mu \delta^4(x-x')$$

$$= \frac{\kappa^2 H^2}{16\pi^2} \times \frac{1}{4} \not{\partial} \delta^4(x-x') + O(D-4) \; . \quad (229)$$

We have summarized the results in Table 17, along with all terms for which the initial contractions of Table 16 produced delta functions. The sum of all such terms is,

$$-i \Bigl[\Sigma^{T17} \Bigr](x;x') = \frac{\kappa^2 H^2}{16\pi^2} \Bigl\{ -\frac{3}{8} \not{\partial} - \frac{1}{4} \overline{\not{\partial}} \Bigr\} \delta^4(x-x') + O(D-4) \; . \quad (230)$$

All the lower vertex pairs involve one or more derivatives of $i\delta\Delta_C$,

$$\partial_i i\delta\Delta_C = -\frac{H^2 \Gamma(\frac{D}{2}-1)}{32\pi^{\frac{D}{2}}} (D-6)(D-4)(aa')^{2-\frac{D}{2}} \frac{\Delta x^i}{\Delta x^{D-2}} = -\partial'_i i\delta\Delta_C \; , \quad (231)$$

Table 17: Delta functions from the $i\delta\Delta_C$ part of the graviton propagator.

I	J	sub	$\frac{\kappa^2 H^2}{16\pi^2} \partial\!\!\!/ \, \delta^4(x-x')$	$\frac{\kappa^2 H^2}{16\pi^2} \overline{\partial\!\!\!/} \, \delta^4(x-x')$
2	1	a	$-\frac{1}{2}$	0
2	1	b	$\frac{1}{4}$	0
2	2	a	$\frac{1}{4}$	0
2	2	b	$-\frac{1}{2}$	$\frac{1}{2}$
2	2	c	$-\frac{1}{8}$	0
2	2	d	$\frac{1}{4}$	$-\frac{1}{4}$
2	3	a	0	0
2	3	b	0	0
Total			$-\frac{3}{8}$	$-\frac{1}{4}$

Table 18: Residual $i\delta\Delta_C$ terms in which all derivatives act upon $\Delta x^2(x;x')$. All contributions are multiplied by $\frac{i\kappa^2 H^2}{2^8 \pi^D} \Gamma(\frac{D}{2}) \Gamma(\frac{D}{2}-1) \frac{(D-4)(D-6)}{D-3} (aa')^{2-\frac{D}{2}}$.

I	J	sub	$\frac{\gamma^\mu \Delta x_\mu}{\Delta x^{2D-2}}$	$\frac{\gamma^i \Delta x_i}{\Delta x^{2D-2}}$	$\frac{\Vert\vec{x}\Vert^2 \gamma^\mu \Delta x_\mu}{\Delta x^{2D}}$	$\frac{\Vert\vec{x}\Vert^2 \gamma^i \Delta x_i}{\Delta x^{2D}}$
2	3	c	$-(D-1)^2$	$D(D-1)$	0	0
2	3	d	0	$(D-1)(D-2)$	$D(D-2)$	$-2D(D-2)$
3	1	a	$4(D-1)$	0	0	0
3	1	b	$-2(D-1)$	$-2(D-4)$	0	0
3	2	a	$-2(D-1)$	0	0	0
3	2	b	$2(D-1)(D-2)$	$-2(D-1)(D-2)$	0	0
3	2	c	$(D-1)$	$(D-4)$	0	0
3	2	d	0	$-(2D-3)(D-2)$	$-2(D-1)(D-2)$	$4(D-1)(D-2)$
3	3	a	$-(D-1)^2$	0	0	0
3	3	b	$\frac{1}{2}(D-1)^2$	$\frac{1}{2}(D-1)(D-4)$	0	0
3	3	c	$\frac{1}{2}(D-1)^2$	$\frac{1}{2}(D-1)(D-4)$	0	0
3	3	d	$-\frac{1}{2}(D-1)(D-2)$	$(D-2)$	$\frac{1}{2}(D-2)^2$	$-(D-2)^2$
Total			$\frac{1}{2}(D-1)(D-2)$	$2(D-1)-D(D-2)$	$-\frac{1}{2}(D-2)^2$	$(D-2)^2$

$$\partial_0 i\delta\Delta_C = \frac{H^2\Gamma(\frac{D}{2}-1)}{32\pi^{\frac{D}{2}}}(D-6)(D-4)(aa')^{2-\frac{D}{2}}\Bigg\{\frac{\Delta\eta}{\Delta x^{D-2}}-\frac{aH}{2\Delta x^{D-4}}\Bigg\}, \quad (232)$$

$$\partial_0' i\delta\Delta_C = \frac{H^2\Gamma(\frac{D}{2}-1)}{32\pi^{\frac{D}{2}}}(D-6)(D-4)(aa')^{2-\frac{D}{2}}\Bigg\{-\frac{\Delta\eta}{\Delta x^{D-2}}-\frac{a'H}{2\Delta x^{D-4}}\Bigg\}. \quad (233)$$

Their reduction follows the same pattern as in the previous two sub-sections. Table 18 summarizes the results for the case in which all derivatives act upon the conformal coordinate separation Δx^2.

When summed, three of the columns of Table 18 reveal a factor of $(D-2)$ which we extract,

$$-i\Big[\Sigma^{T18}\Big](x;x') = \frac{i\kappa^2H^2}{2^8\pi^D}\Gamma\Big(\frac{D}{2}\Big)\Gamma\Big(\frac{D}{2}-1\Big)\frac{(D-2)(D-4)(D-6)}{(D-3)}(aa')^{2-\frac{D}{2}}$$
$$\times\Bigg\{\frac{1}{2}(D-1)\frac{\gamma^\mu\Delta x_\mu}{\Delta x^{2D-2}}+\Big[2\Big(\frac{D-1}{D-2}\Big)-D\Big]\frac{\gamma^i\Delta x_i}{\Delta x^{2D-2}}$$
$$-\frac{1}{2}(D-2)\frac{\|\Delta\vec{x}\|^2\gamma^\mu\Delta x_\mu}{\Delta x^{2D}}+(D-2)\frac{\|\Delta\vec{x}\|^2\gamma^i\Delta x_i}{\Delta x^{2D}}\Bigg\}. \quad (234)$$

We partially integrate Equation 234 with the aid of Equations 193-194 and then take $D=4$, just as we did for the sum of Table 15,

$$-i\Big[\Sigma^{T18}\Big](x;x') = \frac{i\kappa^2H^2}{2^8\pi^D}\Gamma\Big(\frac{D}{2}\Big)\Gamma\Big(\frac{D}{2}-1\Big)\frac{(D-2)(D-4)(D-6)}{(D-3)}(aa')^{2-\frac{D}{2}}$$
$$\times\Bigg\{\frac{D}{4}\frac{\gamma^\mu\Delta x_\mu}{\Delta x^{2D-2}}+\Big[2\Big(\frac{D-1}{D-2}\Big)-\frac{D^2}{2(D-1)}\Big]\frac{\gamma^i\Delta x_i}{\Delta x^{2D-2}}$$
$$-\frac{\nabla^2}{8(D-1)}\Big(\frac{\gamma^\mu\Delta x_\mu}{\Delta x^{2D-4}}\Big)+\frac{\nabla^2}{4(D-1)}\Big(\frac{\gamma^i\Delta x_i}{\Delta x^{2D-4}}\Big)\Bigg\}, \quad (235)$$
$$= \frac{i\kappa^2H^2}{2^8\pi^D}\Gamma\Big(\frac{D}{2}\Big)\Gamma\Big(\frac{D}{2}-1\Big)\frac{(D-2)(D-6)}{(D-1)(D-3)^2}(aa')^{2-\frac{D}{2}}\Bigg\{-\frac{D(D-1)}{16(D-2)}\partial\!\!\!/\partial^2$$
$$+\frac{(D^3-6D^2+8D-4)}{8(D-2)^2}\overline{\partial\!\!\!/}\,\partial^2+\Big(\frac{D-4}{16}\Big)\partial\!\!\!/\nabla^2-\Big(\frac{D-4}{8}\Big)\overline{\partial\!\!\!/}\,\nabla^2\Bigg\}\frac{1}{\Delta x^{2D-6}}, \quad (236)$$
$$= \frac{i\kappa^2H^2}{2^8\pi^4}\Big\{\frac{1}{2}\partial\!\!\!/\,\partial^2+\frac{1}{6}\overline{\partial\!\!\!/}\,\partial^2\Big\}\frac{1}{\Delta x^2}+O(D-4)\;. \quad (237)$$

As already explained, terms for which one or more derivative acts upon a scale factor make no contribution in $D=4$ dimensions, so this is the final nonzero contribution.

3.7 Renormalized Result

The regulated result we have worked so hard to compute derives from summing expressions 127, 141, 158, 183, 196, 199, 221, 230 and 237,

$$
\begin{aligned}
-i\Big[\Sigma\Big](x;x') = \kappa^2 \Big\{ & \beta_1 (aa')^{1-\frac{D}{2}} \bar{\partial}\!\!\!/ \partial^2 + \beta_2 (aa')^{2-\frac{D}{2}} H^2 \partial\!\!\!/ + \beta_3 (aa')^{2-\frac{D}{2}} H^2 \overline{\partial\!\!\!/} \\
& + b_2 H^2 \partial\!\!\!/ + b_3 H^2 \overline{\partial\!\!\!/} \Big\} \delta^D(x-x') + \frac{\kappa^2 H^2}{16\pi^2} \times -3\ln(a) \overline{\partial\!\!\!/} \delta^4(x-x') \\
& - \frac{i\kappa^2}{2^8\pi^4}(aa')^{-1} \partial\!\!\!/ \partial^4 \Big[\frac{\ln(\mu^2\Delta x^2)}{\Delta x^2}\Big] + \frac{i\kappa^2 H^2}{2^8\pi^4} \Big\{ \Big(-\frac{15}{2} \partial\!\!\!/ \partial^2 + \overline{\partial\!\!\!/} \partial^2\Big) \Big[\frac{\ln(\mu^2\Delta x^2)}{\Delta x^2}\Big] \\
& + \Big(8 \overline{\partial\!\!\!/} \partial^2 - 4 \partial\!\!\!/ \nabla^2\Big) \Big[\frac{\ln(\frac{1}{4}H^2\Delta x^2)}{\Delta x^2}\Big] - 7 \partial\!\!\!/ \nabla^2 \Big[\frac{1}{\Delta x^2}\Big] \Big\} + O(D-4). \qquad (238)
\end{aligned}
$$

The various D-dependent constants in Equation 238 are,

$$\beta_1 = \frac{\mu^{D-4}}{2^8 \pi^{\frac{D}{2}}} \frac{\Gamma(\frac{D}{2}-1)}{(D-3)(D-4)} \Big\{ -2D+1-\frac{2}{D-2} \Big\}, \qquad (239)$$

$$\beta_2 = \frac{\mu^{D-4}}{2^9 \pi^{\frac{D}{2}}} \frac{\Gamma(\frac{D}{2}+1)}{(D-3)(D-4)} \Big\{ \frac{1}{2}D^2 - 10D + 15 - \frac{24}{D} - \frac{6}{D-1} - \frac{35}{D-3} \Big\}, \qquad (240)$$

$$\beta_3 = \frac{\mu^{D-4}}{2^9 \pi^{\frac{D}{2}}} \frac{\Gamma(\frac{D}{2}+1)}{(D-3)(D-4)} \Big\{ -D+3+\frac{9}{D-3} \Big\}, \qquad (241)$$

$$
\begin{aligned}
b_2 = \frac{H^{D-4}}{(4\pi)^{\frac{D}{2}}} \frac{\Gamma(D-1)}{\Gamma(\frac{D}{2})} \Big\{ & -\frac{(D+1)(D-1)(D-4)}{2(D-3)} \times \frac{\pi}{2} \cot\Big(\frac{\pi D}{2}\Big) \\
& - \frac{(D-1)(D^3-8D^2+23D-32)}{8(D-2)^2(D-3)^2} - \frac{7}{48} \Big\}, \qquad (242)
\end{aligned}
$$

$$
\begin{aligned}
b_3 = \frac{H^{D-4}}{(4\pi)^{\frac{D}{2}}} \frac{\Gamma(D-1)}{\Gamma(\frac{D}{2})} \Big\{ & \frac{3}{4}\Big(D - \frac{2}{D-3}\Big) \times \frac{\pi}{2} \cot\Big(\frac{\pi D}{2}\Big) \\
& + \frac{3}{4} \frac{(D^2-6D+8)}{(D-2)^2(D-3)^2} - \frac{5}{2} \Big\}. \qquad (243)
\end{aligned}
$$

In obtaining these expressions we have always chosen to convert finite, $D=4$ terms with ∂^2 acting on $1/\Delta x^2$, into delta functions,

$$\partial^2 \Big[\frac{1}{\Delta x^2}\Big] = i4\pi^2 \delta^4(x-x') \; . \qquad (244)$$

Figure 3: Contribution from counterterms.

All such terms have then been included in b_2 and b_3.

The local divergences in this expression are canceled by the BPHZ counterterms enumerated at the end of section 3. The generic diagram topology is depicted in Figure 3, and the analytic form is,

$$
\begin{aligned}
-i\Big[\Sigma^{\rm ctm}\Big](x;x') &= \sum_{I=1}^{3} iC_{Iij}\,\delta^D(x-x')\ , && (245)\\
&= -\kappa^2\Big\{\alpha_1(aa')^{-1}\not{\partial}\partial^2 + \alpha_2 D(D-1)H^2\not{\partial} + \alpha_3 H^2\overline{\not{\partial}}\Big\}\delta^D(x-x'). && (246)
\end{aligned}
$$

In comparing Equation 238 and Equation 246 it would seem that the simplest choice for the coefficients α_i is,

$$
\alpha_1 = \beta_1 \quad , \quad \alpha_2 = \frac{\beta_2+b_2}{D(D-1)} \quad \text{and} \quad \alpha_3 = \beta_3+b_3\ . \tag{247}
$$

This choice absorbs all local constants but one is of course left with time dependent terms proportional to $\ln(aa')$,

$$
\begin{aligned}
\beta_1(aa')^{1-\frac{D}{2}} - \alpha_1(aa')^{-1} &= +\frac{1}{2^6\pi^2}\frac{\ln(aa')}{aa'} + O(D-4)\ , && (248)\\
\beta_2(aa')^{2-\frac{D}{2}} + b_2 - D(D-1)\alpha_2 &= +\frac{7.5}{2^6\pi^2}\ln(aa') + O(D-4)\ , && (249)\\
\beta_3(aa')^{2-\frac{D}{2}} + b_3 - \alpha_3 &= -\frac{1}{2^6\pi^2}\ln(aa') + O(D-4)\ . && (250)
\end{aligned}
$$

Our final result for the renormalized self-energy is,

$$
\begin{aligned}
-i\Big[\Sigma^{\rm ren}\Big](x;x') = {} & \frac{\kappa^2}{2^6\pi^2}\Bigg\{\frac{\ln(aa')}{aa'}\not{\partial}\partial^2 + \frac{15}{2}\ln(aa')H^2\not{\partial} - 7\ln(aa')H^2\overline{\not{\partial}}\Bigg\}\delta^4(x-x')\\
& -\frac{i\kappa^2}{2^8\pi^4}(aa')^{-1}\not{\partial}\partial^4\Big[\frac{\ln(\mu^2\Delta x^2)}{\Delta x^2}\Big] + \frac{i\kappa^2H^2}{2^8\pi^4}\Bigg\{\Big(-\frac{15}{2}\not{\partial}\,\partial^2 + \overline{\not{\partial}}\,\partial^2\Big)\Big[\frac{\ln(\mu^2\Delta x^2)}{\Delta x^2}\Big]\\
& + \Big(8\overline{\not{\partial}}\partial^2 - 4\not{\partial}\nabla^2\Big)\Big[\frac{\ln(\frac{1}{4}H^2\Delta x^2)}{\Delta x^2}\Big] - 7\not{\partial}\,\nabla^2\Big[\frac{1}{\Delta x^2}\Big]\Bigg\}. \qquad (251)
\end{aligned}
$$

4 CORRECTING THE MODES

It is worth summarizing the conventions used in computing the fermion self-energy. We worked on de Sitter background in conformal coordinates,

$$ds^2 = a^2(\eta)\Big(-d\eta^2 + d\vec{x}\cdot d\vec{x}\Big) \qquad \text{where} \qquad a(\eta) = -\frac{1}{H\eta} = e^{Ht} \; . \tag{252}$$

We used dimensional regularization and obtained the self-energy for the conformally re-scaled fermion field,

$$\Psi(x) \equiv a^{(\frac{D-1}{2})}\psi(x) \; . \tag{253}$$

The local Lorentz gauge was fixed to allow an algebraic expression for the vierbein in terms of the metric [40]. The general coordinate gauge was fixed to make the tensor structure of the graviton propagator decouple from its spacetime dependence [41, 50]. The result we obtained is,

$$\begin{aligned}\Big[\Sigma^{\rm ren}\Big](x;x') = \frac{i\kappa^2 H^2}{2^6\pi^2}\Big\{\frac{\ln(aa')}{H^2 aa'}\partial\!\!\!/\partial^2 + \frac{15}{2}\ln(aa')\partial\!\!\!/ - 7\ln(aa')\overline{\partial\!\!\!/}\Big\}\delta^4(x-x') \\ + \frac{\kappa^2}{2^8\pi^4}(aa')^{-1}\partial\!\!\!/\partial^4\Big[\frac{\ln(\mu^2\Delta x^2)}{\Delta x^2}\Big] + \frac{\kappa^2 H^2}{2^8\pi^4}\Big\{\Big(\frac{15}{2}\partial\!\!\!/\,\partial^2 - \overline{\partial\!\!\!/}\,\partial^2\Big)\Big[\frac{\ln(\mu^2\Delta x^2)}{\Delta x^2}\Big] \\ + \Big(-8\overline{\partial\!\!\!/}\partial^2 + 4\partial\!\!\!/\nabla^2\Big)\Big[\frac{\ln(\frac{1}{4}H^2\Delta x^2)}{\Delta x^2}\Big] + 7\partial\!\!\!/\,\nabla^2\Big[\frac{1}{\Delta x^2}\Big]\Big\} + O(\kappa^4).\end{aligned} \tag{254}$$

where $\kappa^2 \equiv 16\pi G$ is the loop counting parameter of quantum gravity. The various differential and spinor-differential operators are,

$$\partial^2 \equiv \eta^{\mu\nu}\partial_\mu\partial_\nu \;\;,\;\; \nabla^2 \equiv \partial_i\partial_i \;\;,\;\; \partial\!\!\!/ \equiv \gamma^\mu\partial_\mu \;\text{ and }\; \overline{\partial\!\!\!/} \equiv \gamma^i\partial_i \; , \tag{255}$$

where $\eta^{\mu\nu}$ is the Lorentz metric and γ^μ are the gamma matrices. The conformal coordinate interval is basically $\Delta x^2 \equiv (x-x')^\mu(x-x')^\nu\eta_{\mu\nu}$, up to a subtlety about the imaginary part which will be explained shortly.

The linearized, effective Dirac equation we will solve is,

$$i\,\partial\!\!\!/_{ij}\Psi_j(x) - \int d^4x'\,\Big[{}_i\Sigma_j\Big](x;x')\,\Psi_j(x') = 0 \; . \tag{256}$$

In judging the validity of this exercise it is important to answer five questions:

1. How do solutions to Equation 256 depend upon the finite parts of counterterms?

2. What is the imaginary part of Δx^2?

3. What can we do without the higher loop contributions to the fermion self-energy?

4. What is the relation between the $\mathbb{C}$-number, effective field Equation 256 and the Heisenberg operator equations of Dirac + Einstein? and

5. How do solutions to Equation 256 change when different gauges are used?

In next section we will comment on issues 1-3. Issues 4 and 5 are closely related, and require a lengthy digression that we have consigned to section 2 of this chapter.

4.1 The Linearized Effective Dirac Equation

Dirac + Einstein is not perturbatively renormalizable [18], so we could only obtain a finite result by absorbing divergences in the BPHZ sense [19, 20, 21, 22] using three counterterms involving either higher derivatives or the curvature $R = 12H^2$,

$$-\kappa^2 H^2 \Big\{ \frac{\alpha_1}{H^2 a a'} \partial\!\!\!/ \partial^2 + \alpha_2 D(D-1) \partial\!\!\!/ + \alpha_3 \overline{\partial\!\!\!/} \Big\} \delta^D(x-x') \; . \tag{257}$$

No physical principle seems to fix the finite parts of these counterterms so any result which derives from their values is arbitrary. We chose to null local terms at the beginning of inflation ($a = 1$), but any other choice could have been made and would have affected the solution to Equation 256. Hence there is no point in solving the equation exactly. However, each of the three counterterms is related to a term in Equation 251 which carries a factor of $\ln(aa')$,

$$\frac{\alpha_1}{H^2 a a'} \partial\!\!\!/ \partial^2 \Longleftrightarrow \frac{\ln(aa')}{H^2 a a'} \partial\!\!\!/ \partial^2 \; , \tag{258}$$

$$\alpha_2 D(D-1) \partial\!\!\!/ \Longleftrightarrow \frac{15}{2} \ln(aa') \partial\!\!\!/ \; , \tag{259}$$

$$\alpha_3 \overline{\partial\!\!\!/} \Longleftrightarrow -7 \ln(aa') \overline{\partial\!\!\!/} \; . \tag{260}$$

Unlike the α_i's, the numerical coefficients of the right hand terms are uniquely fixed and completely independent of renormalization. The factors of $\ln(aa')$

on these right hand terms mean that they dominate over any finite change in the α_i's at late times. It is in this late time regime that we can make reliable predictions about the effect of quantum gravitational corrections.

The analysis we have just made is a standard feature of low energy effective field theory, and has many distinguished antecedents [23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37]. Loops of massless particles make finite, nonanalytic contributions which cannot be changed by counterterms and which dominate the far infrared. Further, these effects must occur as well, with precisely the same numerical values, in whatever fundamental theory ultimately resolves the ultraviolet problems of quantum gravity.

We must also clarify what is meant by the conformal coordinate interval $\Delta x^2(x;x')$ which appears in Equation 251. The in-out effective field equations correspond to the replacement,

$$\Delta x^2(x;x') \longrightarrow \Delta x^2_{++}(x;x') \equiv \|\vec{x}-\vec{x}'\|^2 - (|\,\eta-\eta'\,| - i\delta)^2 \; . \tag{261}$$

These equations govern the evolution of quantum fields under the assumption that the universe begins in free vacuum at asymptotically early times and ends up the same way at asymptotically late times. This is valid for scattering in flat space but not for cosmological settings in which particle production prevents the in vacuum from evolving to the out vacuum. Persisting with the in-out effective field equations would result in quantum correction terms which are dominated by events from the infinite future! This is the correct answer to the question being asked, which is, "what must the field be in order to make the universe to evolve from in vacuum to out vacuum?" However, that question is not very relevant to any observation we can make.

A more realistic question is, "what happens when the universe is released from a prepared state at some finite time and allowed to evolve as it will?" This sort of question can be answered using the Schwinger-Keldysh formalism [74, 75, 76, 77, 78, 79, 80, 81]. Here we digress to briefly derive it. To sketch the derivation, consider a real scalar field, $\varphi(x)$ whose Lagrangian (not Lagrangian density) at time t is $L[\varphi(t)]$. The well-known functional integral expression for the matrix element of an operator $\mathcal{O}_1[\varphi]$ between states whose wave functionals are given at a starting time s and a last time ℓ is

$$\Big\langle \Phi \Big| T^*\Big(\mathcal{O}_1[\varphi]\Big) \Big| \Psi \Big\rangle = \int [d\varphi]\, \mathcal{O}_1[\varphi]\, \Phi^*[\varphi(\ell)]\, e^{i \int_s^\ell dt L[\varphi(t)]}\, \Psi[\varphi(s)] \; . \tag{262}$$

The T^*-ordering symbol in the matrix element indicates that the operator $\mathcal{O}_1[\varphi]$ is time-ordered, except that any derivatives are taken *outside* the time-

ordering. We can use Equation 262 to obtain a similar expression for the matrix element of the *anti*-time-ordered product of some operator $\mathcal{O}_2[\varphi]$ in the presence of the reversed states,

$$\Big\langle \Psi \Big| \overline{T}^* \Big(\mathcal{O}_2[\varphi] \Big) \Big| \Phi \Big\rangle = \Big\langle \Phi \Big| T^* \Big(\mathcal{O}_2^\dagger[\varphi] \Big) \Big| \Psi \Big\rangle^* , \tag{263}$$

$$= \int [d\varphi] \, \mathcal{O}_2[\varphi] \, \Phi[\varphi(\ell)] \, e^{-i \int_s^\ell dt L[\varphi(t)]} \, \Psi^*[\varphi(s)] \; . \tag{264}$$

Now note that summing over a complete set of states Φ gives a delta functional,

$$\sum_\Phi \Phi\Big[\varphi_-(\ell)\Big] \, \Phi^*\Big[\varphi_+(\ell)\Big] = \delta\Big[\varphi_-(\ell) - \varphi_+(\ell)\Big] \; . \tag{265}$$

Taking the product of Equation 262 and Equation 264, and using Equation 265, we obtain a functional integral expression for the expectation value of any anti-time-ordered operator $\mathcal{O}_2$ multiplied by any time-ordered operator $\mathcal{O}_1$,

$$\Big\langle \Psi \Big| \overline{T}^* \Big(\mathcal{O}_2[\varphi] \Big) T^* \Big(\mathcal{O}_1[\varphi] \Big) \Big| \Psi \Big\rangle = \int [d\varphi_+][d\varphi_-] \, \delta\Big[\varphi_-(\ell) - \varphi_+(\ell)\Big]$$

$$\times \mathcal{O}_2[\varphi_-] \mathcal{O}_1[\varphi_+] \Psi^*[\varphi_-(s)] e^{i \int_s^\ell dt \Big\{ L[\varphi_+(t)] - L[\varphi_-(t)] \Big\}} \, \Psi[\varphi_+(s)] \; . \tag{266}$$

This is the fundamental relation between the canonical operator formalism and the functional integral formalism in the Schwinger-Keldysh formalism.

The Feynman rules follow from Equation 266 in close analogy to those for in-out matrix elements. Because the same field is represented by two different dummy functional variables, $\varphi_\pm(x)$, the endpoints of lines carry a $\pm$ polarity. External lines associated with the operator $\mathcal{O}_2[\varphi]$ have $-$ polarity whereas those associated with the operator $\mathcal{O}_1[\varphi]$ have $+$ polarity. Interaction vertices are either all $+$ or all $-$. Vertices with $+$ polarity are the same as in the usual Feynman rules whereas vertices with the $-$ polarity have an additional minus sign. Propagators can be $++$, $-+$, $+-$ and $--$.

The four propagators can be read off from the fundamental relation 266 when the free Lagrangian is substituted for the full one. It is useful to denote canonical expectation values in the free theory with a subscript 0. With this convention we see that the $++$ propagator is just the ordinary Feynman propagator,

$$i\Delta_{++}(x;x') = \Big\langle \Omega \Big| T\Big(\varphi(x)\varphi(x') \Big) \Big| \Omega \Big\rangle_0 = i\Delta(x;x') \; . \tag{267}$$

The other cases are simple to read off and to relate to the Feynman propagator,

$$
\begin{aligned}
i\Delta_{-+}(x;x') &= \Big\langle \Omega \Big| \varphi(x)\varphi(x') \Big| \Omega \Big\rangle_0 \\
&= \theta(t-t') i\Delta(x;x') + \theta(t'-t) \Big[i\Delta(x;x') \Big]^*, \qquad (268) \\
i\Delta_{+-}(x;x') &= \Big\langle \Omega \Big| \varphi(x')\varphi(x) \Big| \Omega \Big\rangle_0 \\
&= \theta(t-t') \Big[i\Delta(x;x') \Big]^* + \theta(t'-t) i\Delta(x;x'), \qquad (269) \\
i\Delta_{--}(x;x') &= \Big\langle \Omega \Big| \overline{T}\Big(\varphi(x)\varphi(x') \Big) \Big| \Omega \Big\rangle_0 = \Big[i\Delta(x;x') \Big]^*. \qquad (270)
\end{aligned}
$$

Therefore we can get the four propagators of the Schwinger-Keldysh formalism from the Feynman propagator once that is known.

Because external lines can be either + or − every N-point 1PI function of the in-out formalism gives rise to 2^N 1PI functions in the Schwinger-Keldysh formalism. For example, the 1PI 2-point function of the in-out formalism — which is known as the self-mass-squared $M^2(x;x')$ for our scalar example — generalizes to four self-mass-squared functions,

$$
M^2(x;x') \longrightarrow M^2_{\pm\pm}(x;x') \; . \qquad (271)
$$

The first subscript denotes the polarity of the first position x^μ and the second subscript gives the polarity of the second position x'^μ.

Recall that the in-out effective action is the generating functional of 1PI functions. Hence its expansion in powers of the background field $\phi(x)$ takes the form,

$$
\Gamma[\phi] = S[\phi] - \frac{1}{2} \int d^4x \int d^4x' \phi(x) M^2(x;x') \phi(x') + O(\phi^3) \; , \qquad (272)
$$

where $S[\phi]$ is the classical action. In contrast, the Schwinger-Keldysh effective action must depend upon two fields — call them $\phi_+(x)$ and $\phi_-(x)$ — in order to access the different polarities. At lowest order in the weak field expansion we have,

$$
\begin{aligned}
\Gamma[\phi_+,\phi_-] &= S[\phi_+] - S[\phi_-] \\
&\quad -\frac{1}{2} \int d^4x \int d^4x' \Big\{ \phi_+(x) M^2_{++}(x;x') \phi_+(x') + \phi_+(x) M^2_{+-}(x;x') \phi_-(x') \\
&\quad + \phi_-(x) M^2_{-+}(x;x') \phi_+(x') + \phi_-(x) M^2_{--}(x;x') \phi_-(x') \Big\} + O(\phi^3_\pm). \qquad (273)
\end{aligned}
$$

The effective field equations of the in-out formalism are obtained by varying the in-out effective action,

$$\frac{\delta\Gamma[\phi]}{\delta\phi(x)} = \frac{\delta S[\phi]}{\delta\phi(x)} - \int d^4x' M^2(x;x')\phi(x') + O(\phi^2). \tag{274}$$

Note that these equations are not causal in the sense that the integral over x'^μ receives contributions from points to the future of x^μ. No initial value formalism is possible for these equations. Note also that even a Hermitian field operator such as $\varphi(x)$ will not generally admit purely real effective field solutions $\phi(x)$ because 1PI functions have imaginary parts. This makes the in-out effective field equations quite unsuitable for applications in cosmology.

The Schwinger-Keldysh effective field equations are obtained by varying with respect to ϕ_+ and then setting both fields equal,

$$\frac{\delta\Gamma[\phi_+,\phi_-]}{\delta\phi_+(x)}\Bigg|_{\phi_\pm=\phi} = \frac{\delta S[\phi]}{\delta\phi(x)} - \int d^4x' \Big[M^2_{++}(x;x') + M^2_{+-}(x;x')\Big]\phi(x') + O(\phi^2). \tag{275}$$

The sum of $M^2_{++}(x;x')$ and $M^2_{+-}(x;x')$ is zero unless x'^μ lies on or within the past light-cone of x^μ. So the Schwinger-Keldysh field equations admit a well-defined initial value formalism in spite of the fact that they are nonlocal. Note also that the sum of $M^2_{++}(x;x')$ and $M^2_{+-}(x;x')$ is real, which neither 1PI function is separately.

From the preceding discussion we can infer these simple rules:

- The linearized effective Dirac equation of the Schwinger-Keldysh formalism takes the form Equation 256 with the replacement,

$$\Big[{}_i\Sigma_j\Big](x;x') \longrightarrow \Big[{}_i\Sigma_j\Big]_{++}(x;x') + \Big[{}_i\Sigma_j\Big]_{+-}(x;x') \; ; \tag{276}$$

- The $_{++}$ fermion self-energy is Equation 251 with the replacement Equation 261; and

- The $_{+-}$ fermion self-energy is,

$$-\frac{\kappa^2}{2^8\pi^4 aa'}\partial\!\!\!/\partial^4\Big[\frac{\ln(\mu^2\Delta x^2)}{\Delta x^2}\Big] - \frac{\kappa^2H^2}{2^8\pi^4}\Bigg\{\Big(\frac{15}{2}\partial\!\!\!/\,\partial^2 - \overline{\partial\!\!\!/}\,\partial^2\Big)\Big[\frac{\ln(\mu^2\Delta x^2)}{\Delta x^2}\Big] \\ + \Big(-8\overline{\partial\!\!\!/}\partial^2 + 4\partial\!\!\!/\nabla^2\Big)\Big[\frac{\ln(\frac14 H^2\Delta x^2)}{\Delta x^2}\Big] + 7\partial\!\!\!/\,\nabla^2\Big[\frac{1}{\Delta x^2}\Big]\Bigg\} + O(\kappa^4), \tag{277}$$

  with the replacement,

$$\Delta x^2(x;x') \longrightarrow \Delta x^2_{+-}(x;x') \equiv \|\vec{x} - \vec{x}'\|^2 - (\eta - \eta' + i\delta)^2 \; . \tag{278}$$

The difference of the $_{++}$ and $_{+-}$ terms leads to zero contribution in Equation 256 unless the point x'^{μ} lies on or within the past light-cone of x^{μ}.

We can only solve for the one loop corrections to the field because we lack the higher loop contributions to the self-energy. The general perturbative expansion takes the form,

$$\Psi(x) = \sum_{\ell=0}^{\infty} \kappa^{2\ell} \Psi^{\ell}(x) \qquad \text{and} \quad \Big[\Sigma\Big](x;x') = \sum_{\ell=1}^{\infty} \kappa^{2\ell} \Big[\Sigma^{\ell}\Big](x;x') \; . \tag{279}$$

One substitutes these expansions into the effective Dirac equation in Equation 256 and then segregates powers of κ^2,

$$i\partial\!\!\!/ \Psi^0(x) = 0 \qquad , \qquad i\partial\!\!\!/ \Psi^1(x) = \int d^4x' \Big[\Sigma^1\Big](x;x')\Psi^0(x') \qquad \text{et cetera.} \tag{280}$$

We shall work out the late time limit of the one loop correction $\Psi^1_i(\eta,\vec{x};\vec{k},s)$ for a spatial plane wave of helicity s,

$$\Psi^0_i(\eta,\vec{x};\vec{k},s) = \frac{e^{-ik\eta}}{\sqrt{2k}} u_i(\vec{k},s) e^{i\vec{k}\cdot\vec{x}} \qquad \text{where} \qquad k^{\ell}\gamma^{\ell}_{ij} u_j(\vec{k},s) = k\gamma^0_{ij} u_j(\vec{k},s) \; . \tag{281}$$

4.2 Effective Field Equations

The purpose of this section is to elucidate the relation between the Heisenberg operators of Dirac + Einstein — $\overline{\psi}_i(x)$, $\psi_i(x)$ and $h_{\mu\nu}(x)$ — and the $\mathbb{C}$-number plane wave mode solutions $\Psi_i(x;\vec{k},s)$ of the linearized, effective Dirac equation in Equation 256. After explaining the relation we work out an example, at one loop order, in a simple scalar analogue model. Finally, we return to Dirac + Einstein to explain how $\Psi_i(x;\vec{k},s)$ changes with variations of the gauge.

One solves the gauge-fixed Heisenberg operator equations perturbatively,

$$h_{\mu\nu}(x) = h^0_{\mu\nu}(x) + \kappa h^1_{\mu\nu}(x) + \kappa^2 h^2_{\mu\nu}(x) + \dots \; , \tag{282}$$

$$\psi_i(x) = \psi^0_i(x) + \kappa\psi^1_i(x) + \kappa^2\psi^2_i(x) + \dots \; . \tag{283}$$

Because our state is released in free vacuum at $t = 0$ ($\eta = -1/H$), it makes sense to express the operator as a functional of the creation and annihilation operators of this free state. So our initial conditions are that $h_{\mu\nu}$ and its

first time derivative coincide with those of $h^0_{\mu\nu}(x)$ at $t = 0$, and also that $\psi_i(x)$ coincides with $\psi^0_i(x)$. The zeroth order solutions to the Heisenberg field equations take the form,

$$\begin{aligned} h^0_{\mu\nu}(x) &= \int \frac{d^{D-1}k}{(2\pi)^{D-1}} \sum_\lambda \Bigl\{\epsilon_{\mu\nu}(\eta;\vec{k},\lambda) e^{i\vec{k}\cdot\vec{x}} \alpha(\vec{k},\lambda) \\ &\qquad + \epsilon^*_{\mu\nu}(\eta;\vec{k},\lambda) e^{-i\vec{k}\cdot\vec{x}} \alpha^\dagger(\vec{k},\lambda)\Bigr\} , \quad (284) \\ \psi^0_i(x) &= a^{-(\frac{D-1}{2})} \int \frac{d^{D-1}k}{(2\pi)^{D-1}} \sum_s \Bigl\{\frac{e^{-ik\eta}}{\sqrt{2k}} u_i(\vec{k},s) e^{i\vec{k}\cdot\vec{x}} b(\vec{k},s) \\ &\qquad + \frac{e^{ik\eta}}{\sqrt{2k}} v_i(\vec{k},\lambda) e^{-i\vec{k}\cdot\vec{x}} c^\dagger(\vec{k},s)\Bigr\} . \quad (285) \end{aligned}$$

The graviton mode functions are proportional to Hankel functions whose precise specification we do not require. The Dirac mode functions $u_i(\vec{k},s)$ and $v_i(\vec{k},s)$ are precisely those of flat space by virtue of the conformal invariance of massless fermions. The canonically normalized creation and annihilation operators obey,

$$\Bigl[\alpha(\vec{k},\lambda), \alpha^\dagger(\vec{k}',\lambda')\Bigr] = \delta_{\lambda\lambda'} (2\pi)^{D-1} \delta^{D-1}(\vec{k}-\vec{k}') , \quad (286)$$

$$\Bigl\{b(\vec{k},s), b^\dagger(\vec{k}',s')\Bigr\} = \delta_{ss'} (2\pi)^{D-1} \delta^{D-1}(\vec{k}-\vec{k}') = \Bigl\{c(\vec{k},s), c^\dagger(\vec{k}',s')\Bigr\} . \quad (287)$$

The zeroth order Fermi field $\psi^0_i(x)$ is an anti-commuting operator whereas the mode function $\Psi^0(x;\vec{k},s)$ is a $\mathbb{C}$-number. The latter can be obtained from the former by anti-commuting with the fermion creation operator,

$$\Psi^0_i(x;\vec{k},s) = a^{\frac{D-1}{2}} \Bigl\{\psi^0_i(x), b^\dagger(\vec{k},s)\Bigr\} = \frac{e^{-ik\eta}}{\sqrt{2k}} u_i(\vec{k},s) e^{i\vec{k}\cdot\vec{x}} . \quad (288)$$

The higher order contributions to $\psi_i(x)$ are no longer linear in the creation and annihilation operators, so anti-commuting the full solution $\psi_i(x)$ with $b^\dagger(\vec{k},s)$ produces an operator. The quantum-corrected fermion mode function we obtain by solving Equation 256 is the expectation value of this operator in the presence of the state which is free vacuum at $t = 0$,

$$\Psi_i(x;\vec{k},s) = a^{\frac{D-1}{2}} \Bigl\langle\Omega\Bigl|\Bigl\{\psi_i(x), b^\dagger(\vec{k},s)\Bigr\}\Bigr|\Omega\Bigr\rangle . \quad (289)$$

This is what the Schwinger-Keldysh field equations give. The more familiar, in-out effective field equations obey a similar relation except that one defines the free fields to agree with the full ones in the asymptotic past, and one takes the in-out matrix element after anti-commuting.

4.3 A Worked-Out Example

It is perhaps worth seeing a worked-out example, at one loop order, of the relation 289 between the Heisenberg operators and the Schwinger-Keldysh field equations. To simplify the analysis we will work with a model of two scalars in flat space,

$$\mathcal{L} = -\partial_\mu \varphi^* \partial^\mu \varphi - m^2 \varphi^* \varphi - \lambda \chi :\! \varphi^* \varphi \!: -\frac{1}{2} \partial_\mu \chi \partial^\mu \chi \; . \tag{290}$$

In this model φ plays the role of our fermion ψ_i, and χ plays the role of the graviton $h_{\mu\nu}$. Note that we have normal-ordered the interaction term to avoid the harmless but time-consuming digression that would be required to deal with χ developing a nonzero expectation value. We shall also omit discussion of counterterms.

The Heisenberg field equations for Equation 290 are,

$$\partial^2 \chi - \lambda :\! \varphi^* \varphi \!: \; = \; 0 \; , \tag{291}$$

$$(\partial^2 - m^2) \varphi - \lambda \chi \varphi \; = \; 0 \; . \tag{292}$$

As with Dirac + Einstein, we solve these equations perturbatively,

$$\chi(x) \; = \; \chi^0(x) + \lambda \chi^1(x) + \lambda^2 \chi^2(x) + \dots \; , \tag{293}$$

$$\varphi(x) \; = \; \varphi^0(x) + \lambda \varphi^1(x) + \lambda^2 \varphi^2(x) + \dots \; . \tag{294}$$

The zeroth order solutions are,

$$\chi^0(x) \; = \; \int \frac{d^{D-1}k}{(2\pi)^{D-1}} \Big\{ \frac{e^{-ikt}}{\sqrt{2k}} e^{i\vec{k}\cdot\vec{x}} \alpha(\vec{k}) + \frac{e^{ikt}}{\sqrt{2k}} e^{-i\vec{k}\cdot\vec{x}} \alpha^\dagger(\vec{k}) \Big\} \; , \tag{295}$$

$$\varphi^0(x) \; = \; \int \frac{d^{D-1}k}{(2\pi)^{D-1}} \Big\{ \frac{e^{-i\omega t}}{\sqrt{2\omega}} e^{i\vec{k}\cdot\vec{x}} b(\vec{k}) + \frac{e^{i\omega t}}{\sqrt{2\omega}} e^{-i\vec{k}\cdot\vec{x}} c^\dagger(\vec{k}) \Big\} \; . \tag{296}$$

Here $k \equiv \|\vec{k}\|$ and $\omega \equiv \sqrt{k^2 + m^2}$. The creation and annihilation operators are canonically normalized,

$$\Big[\alpha(\vec{k}), \alpha^\dagger(\vec{k}')\Big] = \Big[b(\vec{k}), b^\dagger(\vec{k}')\Big] = \Big[c(\vec{k}), c^\dagger(\vec{k}')\Big] = (2\pi)^{D-1} \delta^{D-1}(\vec{k} - \vec{k}') \; . \tag{297}$$

We choose to develop perturbation theory so that all the operators and their first time derivatives agree with the zeroth order solutions at $t = 0$. The first

few higher order terms are,

$$\chi^1(x)=\int_0^t dt'\int d^{D-1}x' \left\langle x\Big|\frac{1}{\partial^2}\Big|x'\right\rangle_{\rm ret} :\varphi^{0*}(x')\varphi^0(x'):, \tag{298}$$

$$\varphi^1(x)=\int_0^t dt'\int d^{D-1}x' \left\langle x\Big|\frac{1}{\partial^2-m^2}\Big|x'\right\rangle_{\rm ret} \chi^0(x')\varphi^0(x') , \tag{299}$$

$$\varphi^2(x)=\int_0^t dt'\int d^{D-1}x' \left\langle x\Big|\frac{1}{\partial^2-m^2}\Big|x'\right\rangle_{\rm ret} \Big\{\chi^1(x')\varphi^0(x')+\chi^0(x')\varphi^1(x')\Big\}. \tag{300}$$

The commutator of $\varphi^0(x)$ with $b^\dagger(\vec{k})$ is a $\mathbb{C}$-number,

$$\Big[\varphi^0(x), b^\dagger(\vec{k})\Big] = \frac{e^{-i\omega t}}{\sqrt{2\omega}}\, e^{i\vec{k}\cdot\vec{x}} \equiv \Phi^0(x;\vec{k}) . \tag{301}$$

However, commuting the full solution with $b^\dagger(\vec{k})$ leaves operators,

$$\begin{aligned}\Big[\varphi(x), b^\dagger(\vec{k})\Big] &= \Phi^0(x;\vec{k}) + \lambda \int_0^t dt'\int d^{D-1}x' \left\langle x\Big|\frac{1}{\partial^2-m^2}\Big|x'\right\rangle_{\rm ret} \chi^0(x')\Phi^0(x';\vec{k}) \\ &+\lambda^2 \int_0^t dt'\int d^{D-1}x' \left\langle x\Big|\frac{1}{\partial^2-m^2}\Big|x'\right\rangle_{\rm ret} \Bigg\{\Big[\chi^1(x'), b^\dagger(\vec{k})\Big]\varphi^0(x') + \chi^1(x')\Phi^0(x';\vec{k}) \\ &\qquad +\chi^0(x')\Big[\varphi^1(x'), b^\dagger(\vec{k})\Big]\Bigg\} + O(\lambda^3) .\end{aligned} \tag{302}$$

The commutators in Equation 302 are easily evaluated,

$$\begin{aligned}&\Big[\chi^1(x'), b^\dagger(\vec{k})\Big]\varphi^0(x') \\ &\qquad = \int_0^{t'} dt''\int d^{D-1}x'' \left\langle x'\Big|\frac{1}{\partial^2}\Big|x''\right\rangle_{\rm ret} \varphi^{0*}(x'')\varphi^0(x')\Phi^0(x'';\vec{k}) ,\end{aligned} \tag{303}$$

$$\begin{aligned}&\chi^0(x')\Big[\varphi^1(x'), b^\dagger(\vec{k})\Big] \\ &\qquad = \int_0^{t'} dt''\int d^{D-1}x'' \left\langle x'\Big|\frac{1}{\partial^2-m^2}\Big|x''\right\rangle_{\rm ret} \chi^0(x')\chi^0(x'')\Phi^0(x'';\vec{k}) .\end{aligned} \tag{304}$$

Hence the expectation value of Equation 302 gives,

$$\begin{aligned}\Big\langle\Omega\Big|\Big[\varphi(x), b^\dagger(\vec{k})\Big]\Big|\Omega\Big\rangle &= \Phi^0(x;\vec{k}) + \lambda^2 \int_0^t dt'\int d^{D-1}x' \left\langle x\Big|\frac{1}{\partial^2-m^2}\Big|x'\right\rangle_{\rm ret} \\ &\times \int_0^{t'} dt''\int d^{D-1}x'' \Bigg\{\left\langle x'\Big|\frac{1}{\partial^2}\Big|x''\right\rangle_{\rm ret} \Big\langle\Omega\Big|\varphi^{0*}(x'')\varphi^0(x')\Big|\Omega\Big\rangle \\ &\qquad +\left\langle x'\Big|\frac{1}{\partial^2-m^2}\Big|x''\right\rangle_{\rm ret} \Big\langle\Omega\Big|\chi^0(x')\chi^0(x'')\Big|\Omega\Big\rangle\Bigg\}\Phi^0(x'';\vec{k}) + O(\lambda^4).\end{aligned} \tag{305}$$

To make contact with the effective field equations we must first recognize that the retarded Green's functions can be written in terms of expectation values of the free fields,

$$\Big\langle x'\Big|\frac{1}{\partial^2}\Big|x''\Big\rangle_{\rm ret} = -i\theta(t'-t'')\Big[\chi^0(x'),\chi^0(x'')\Big] \tag{306}$$

$$= -i\theta(t'-t\prime\prime)\Big\{\Big\langle\Omega\Big|\chi^0(x')\chi^0(x'')\Big|\Omega\Big\rangle - \Big\langle\Omega\Big|\chi^0(x'')\chi^0(x')\Big|\Omega\Big\rangle\Big\}\ , \tag{307}$$

$$\Big\langle x'\Big|\frac{1}{\partial^2-m^2}\Big|x''\Big\rangle_{\rm ret} = -i\theta(t'-t'')\Big[\varphi^0(x'),\varphi^{0*}(x'')\Big] \tag{308}$$

$$= -i\theta(t'-t'')\Big\{\Big\langle\Omega\Big|\varphi^0(x')\varphi^{0*}(x'')\Big|\Omega\Big\rangle - \Big\langle\Omega\Big|\varphi^{*0}(x'')\varphi^0(x')\Big|\Omega\Big\rangle\Big\}. \tag{309}$$

Substituting these relations into Equation 305 and canceling some terms gives the expression we have been seeking,

$$\Big\langle\Omega\Big|\Big[\varphi(x),b^\dagger(\vec{k})\Big]\Big|\Omega\Big\rangle = \Phi^0(x;\vec{k}) - i\lambda^2\int_0^t dt'\int d^{D-1}x'\,\Big\langle x\Big|\frac{1}{\partial^2-m^2}\Big|x'\Big\rangle_{\rm ret}$$
$$\times\int_0^{t'}dt''\int d^{D-1}x''\,\Big\{\Big\langle\Omega\Big|\chi^0(x')\chi^0(x'')\Big|\Omega\Big\rangle\Big\langle\Omega\Big|\varphi^0(x')\varphi^{0*}(x'')\Big|\Omega\Big\rangle$$
$$-\Big\langle\Omega\Big|\chi^0(x'')\chi^0(x')\Big|\Omega\Big\rangle\Big\langle\Omega\Big|\varphi^{0*}(x'')\varphi^0(x')\Big|\Omega\Big\rangle\Big\}\Phi^0(x'';\vec{k}) + O(\lambda^4). \tag{310}$$

We turn now to the effective field equations of the Schwinger-Keldysh formalism. The $\mathbb{C}$-number field corresponding to $\varphi(x)$ at linearized order is $\Phi(x)$. If the state is released at $t=0$ then the equation $\Phi(x)$ obeys is,

$$(\partial^2-m^2)\Phi(x) - \int_0^t dt'\int d^{D-1}x'\Big\{M^2_{++}(x;x') + M^2_{+-}(x;x')\Big\}\Phi(x') = 0\ . \tag{311}$$

The one loop diagram for the self-mass-squared of φ is depicted in Figure 4.

Because the self-mass-squared has two external lines, there are $2^2 = 4$ polarities in the Schwinger-Keldysh formalism. The two we require are [15, 81],

$$-iM^2_{++}(x;x') = (-i\lambda)^2\Big\langle x\Big|\frac{i}{\partial^2}\Big|x'\Big\rangle_{++}\Big\langle x\Big|\frac{i}{\partial^2-m^2}\Big|x'\Big\rangle_{++} + O(\lambda^4)\ , \tag{312}$$

$$-iM^2_{+-}(x;x') = (-i\lambda)(+i\lambda)\Big\langle x\Big|\frac{i}{\partial^2}\Big|x'\Big\rangle_{+-}\Big\langle x\Big|\frac{i}{\partial^2-m^2}\Big|x'\Big\rangle_{+-} + O(\lambda^4)\ . \tag{313}$$

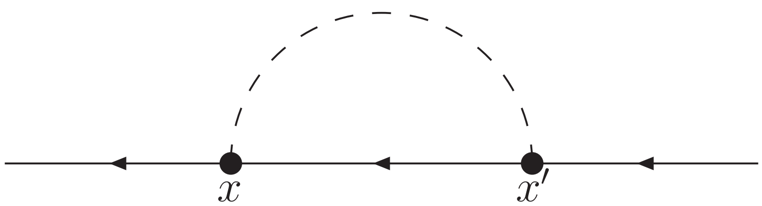

Figure 4: Self-mass-squared for φ at one loop order. Solid lines stands for φ propagators while dashed lines represent χ propagators.

To recover Equation 310 we must express the various Schwinger-Keldysh propagators in terms of expectation values of the free fields. The $_{++}$ polarity gives the usual Feynman propagator [81],

$$\Big\langle x \Big| \frac{i}{\partial^2} \Big| x' \Big\rangle_{++} = \theta(t-t') \Big\langle \Omega \Big| \chi^0(x) \chi^0(x') \Big| \Omega \Big\rangle + \theta(t'-t) \Big\langle \Omega \Big| \chi^0(x') \chi^0(x) \Big| \Omega \Big\rangle \; , \tag{314}$$

$$\Big\langle x \Big| \frac{i}{\partial^2 - m^2} \Big| x' \Big\rangle_{++} = \theta(t-t') \Big\langle \Omega \Big| \varphi^0(x) \varphi^{0*}(x') \Big| \Omega \Big\rangle + \theta(t'-t) \Big\langle \Omega \Big| \varphi^{0*}(x') \varphi^0(x) \Big| \Omega \Big\rangle . \tag{315}$$

The $_{+-}$ polarity propagators are [81],

$$\Big\langle x \Big| \frac{i}{\partial^2} \Big| x' \Big\rangle_{+-} = \Big\langle \Omega \Big| \chi^0(x') \chi^0(x) \Big| \Omega \Big\rangle \; , \tag{316}$$

$$\Big\langle x \Big| \frac{i}{\partial^2 - m^2} \Big| x' \Big\rangle_{+-} = \Big\langle \Omega \Big| \varphi^{0*}(x') \varphi^0(x) \Big| \Omega \Big\rangle \; . \tag{317}$$

Substituting these relations into Equation 312 and Equation 313 and making use of the identity $1 = \theta(t-t') + \theta(t'-t)$ gives,

$$M^2_{++}(x;x') + M^2_{+-}(x;x') = -i\lambda^2 \theta(t-t') \Big\{ \Big\langle \Omega \Big| \chi^0(x) \chi^0(x') \Big| \Omega \Big\rangle \times \Big\langle \Omega \Big| \varphi^0(x) \varphi^{0*}(x') \Big| \Omega \Big\rangle - \Big\langle \Omega \Big| \chi^0(x') \chi^0(x) \Big| \Omega \Big\rangle \Big\langle \Omega \Big| \varphi^{0*}(x') \varphi^0(x) \Big| \Omega \Big\rangle \Big\} + O(\lambda^4) . \tag{318}$$

We now solve Equation 311 perturbatively. The free plane wave mode function 301 is of course a solution at order λ^0. With Equation 318 we easily recognize its perturbative development as,

$$\Phi(x;\vec{k}) = \Phi^0(x;\vec{k}) - i\lambda^2 \int_0^t dt' \int d^{D-1}x' \left\langle x \middle| \frac{1}{\partial^2 - m^2} \middle| x' \right\rangle_{\rm ret}$$
$$\times \int_0^{t'} dt'' \int d^{D-1}x'' \Big\{ \Big\langle \Omega \Big| \chi^0(x')\chi^0(x'') \Big| \Omega \Big\rangle \Big\langle \Omega \Big| \varphi^0(x')\varphi^{0*}(x'') \Big| \Omega \Big\rangle$$
$$- \Big\langle \Omega \Big| \chi^0(x'')\chi^0(x') \Big| \Omega \Big\rangle \Big\langle \Omega \Big| \varphi^{0*}(x'')\varphi^0(x') \Big| \Omega \Big\rangle \Big\} \Phi^0(x'';\vec{k}) + O(\lambda^4). \quad (319)$$

That agrees with Equation 310, so we have established the desired connection,

$$\Phi(x;\vec{k}) = \Big\langle \Omega \Big| \Big[\varphi(x), b^\dagger(\vec{k}) \Big] \Big| \Omega \Big\rangle , \quad (320)$$

at one loop order.

4.4 Gauge Issues

The preceding discussion has made clear that we are working in a particular local Lorentz and general coordinate gauge. We are also doing perturbation theory. The function $\Psi_i^0(x;\vec{k},s)$ describes how a free fermion of wave number $\vec{k}$ and helicity s propagates through classical de Sitter background in our gauge. What $\Psi_i^1(x;\vec{k},s)$ gives is the first quantum correction to this mode function. It is natural to wonder how the effective field $\Psi_i(x;\vec{k},s)$ changes if a different gauge is used.

The operators of the original, invariant Lagrangian transform as follows under diffeomorphisms ($x^\mu \rightarrow x'^\mu$) and local Lorentz rotations (Λ_{ij}),[5]

$$\psi'_i(x) = \Lambda_{ij}\Big(x'^{-1}(x)\Big)\psi_j\Big(x'^{-1}(x)\Big) , \quad (321)$$
$$e'_{\mu b}(x) = \frac{\partial x^\nu}{\partial x'^\mu} \Lambda_b{}^c\Big(x'^{-1}(x)\Big) e_{\nu c}\Big(x'^{-1}(x)\Big) . \quad (322)$$

[5]Of course the spinor and vector representations of the local Lorentz transformation are related as usual, with same parameters $\omega_{cd}(x)$ contracted into the appropriate representation matrices,

$$\Lambda_{ij} \equiv \delta_{ij} - \frac{i}{2}\omega_{cd} J^{cd}{}_{ij} + \dots \qquad \text{and} \qquad \Lambda_b{}^c = \delta_b{}^c - \omega_b{}^c + \dots .$$

The invariance of the theory guarantees that the transformation of any solution is also a solution. Hence the possibility of performing local transformations precludes the existence of a unique initial value solution. This is why no Hamiltonian formalism is possible until the gauge has been fixed sufficiently to eliminate transformations which leave the initial value surface unaffected.

Different gauges can be reached using field-dependent gauge transformations [82]. This has a relatively simple effect upon the Heisenberg operator $\psi_i(x)$, but a complicated one on the linearized effective field $\Psi_i(x;\vec{k},s)$. Because local Lorentz and diffeomorphism gauge conditions are typically specified in terms of the gravitational fields, we assume x'^{μ} and Λ_{ij} depend upon the graviton field $h_{\mu\nu}$. Hence so too does the transformed field,

$$\psi'_i[h](x) = \Lambda_{ij}[h]\Big(x'^{-1}[h](x)\Big)\psi_j\Big(x'^{-1}[h](x)\Big) \; . \qquad (323)$$

In the general case that the gauge changes even on the initial value surface, the creation and annihilation operators also transform,

$$b'[h](\vec{k},s) = \frac{1}{\sqrt{2k}} u^*_i(\vec{k},s) \int d^{D-1}x \, e^{-i\vec{k}\cdot\vec{x}} \psi'_i[h](\eta_i,\vec{x}) \; , \qquad (324)$$

where $\eta_i \equiv -1/H$ is the initial conformal time. Hence the linearized effective field transforms to,

$$\Psi'_i(x;\vec{k},s) = a^{\frac{D-1}{2}} \Big\langle \Omega \Big| \Big\{ \psi'_i[h](x), b'^{\dagger}[h](\vec{k},s) \Big\} \Big| \Omega \Big\rangle \; . \qquad (325)$$

This is quite a complicated relation. Note in particular that the $h_{\mu\nu}$ dependence of $x'^{\mu}[h]$ and $\Lambda_{ij}[h]$ means that $\Psi'_i(x;\vec{k},s)$ is not simply a Lorentz transformation of the original function $\Psi_i(x;\vec{k},s)$ evaluated at some transformed point.

5 ENHANCED FERMION MODE FUNCTION

We first modify our regularized result for the fermion self energy by the employing Schwinger-Keldysh formalism to make it causal and real. We then solve the quantum corrected Dirac equation and find the fermion mode function at late times. Our result is that it grows without bound as if there were a time-dependent field strength renormalization of the free field mode

function. If inflation lasts long enough, perturbation theory must break down. The same result occurs in the Hartree approximation although the numerical coefficients differ.

5.1 Some Key Reductions

The purpose of this section is to derive three results that are used repeatedly in reducing the nonlocal contributions to the effective field equations. We observe that the nonlocal terms of Equation 251 contain $1/\Delta x^2$. We can avoid denominators by extracting another derivative,

$$\frac{1}{\Delta x^2} = \frac{\partial^2}{4} \ln(\Delta x^2) \qquad \text{and} \qquad \frac{\ln(\Delta x^2)}{\Delta x^2} = \frac{\partial^2}{8} \Big[\ln^2(\Delta x^2) - 2\ln(\Delta x^2)\Big] \; . \tag{326}$$

The Schwinger-Keldysh field equations involve the difference of $_{++}$ and $_{+-}$ terms, for example,

$$\frac{\ln(\mu^2 \Delta x^2_{++})}{\Delta x^2_{++}} - \frac{\ln(\mu^2 \Delta x^2_{+-})}{\Delta x^2_{+-}}$$
$$= \frac{\partial^2}{8} \Big\{ \ln^2(\mu^2 \Delta x^2_{++}) - 2\ln(\mu^2 \Delta x^2_{++}) - \ln^2(\mu^2 \Delta x^2_{+-}) + 2\ln(\mu^2 \Delta x^2_{+-}) \Big\} . \tag{327}$$

We now define the coordinate intervals $\Delta\eta \equiv \eta - \eta'$ and $\Delta x \equiv \|\vec{x} - \vec{x}'\|$ in terms of which the $_{++}$ and $_{+-}$ intervals are,

$$\Delta x^2_{++} = \Delta x^2 - (|\Delta\eta| - i\delta)^2 \;\; \text{and} \;\; \Delta x^2_{+-} = \Delta x^2 - (\Delta\eta + i\delta)^2 \; . \tag{328}$$

When $\eta' > \eta$ we have $\Delta x^2_{++} = \Delta x^2_{+-}$, so the $_{++}$ and $_{+-}$ terms in Equation 327 cancel. This means there is no contribution from the future. When $\eta' < \eta$ and $\Delta x > \Delta\eta$ (past spacelike separation) we can take $\delta = 0$,

$$\ln(\mu^2 \Delta x^2_{++}) = \ln[\mu^2(\Delta x^2 - \Delta\eta^2)] = \ln(\mu^2 \Delta x^2_{+-}) \qquad (\Delta x > \Delta\eta > 0) \; . \tag{329}$$

So the $_{++}$ and $_{+-}$ terms again cancel. Only for $\eta' < \eta$ and $\Delta x < \Delta\eta$ (past timelike separation) are the two logarithms different,

$$\ln(\mu^2 \Delta x^2_{+\pm}) = \ln[\mu^2(\Delta\eta^2 - \Delta x^2)] \pm i\pi \qquad (\Delta\eta > \Delta x > 0) \; . \tag{330}$$

Hence Equation 327 can be written as,

$$\frac{\ln(\mu^2 \Delta x^2_{++})}{\Delta x^2_{++}} - \frac{\ln(\mu^2 \Delta x^2_{+-})}{\Delta x^2_{+-}} = \frac{i\pi}{2} \partial^2 \Big\{ \theta(\Delta\eta - \Delta x) \Big[\ln(\mu^2(\Delta\eta^2 - \Delta x^2) - 1\Big] \Big\} . \tag{331}$$

This step shows how the Schwinger-Kledysh formalism achieves causality.

To integrate expression 331 up against the plane wave mode function 281 we first pull the x^μ derivatives outside the integration, then make the change of variables $\vec{x}' = \vec{x} + \vec{r}$ and perform the angular integrals,

$$\int d^4x' \Big\{ \frac{\ln(\mu^2 \Delta x_{++}^2)}{\Delta x_{++}^2} - \frac{\ln(\mu^2 \Delta x_{+-}^2)}{\Delta x_{+-}^2} \Big\} \Psi_i^0(\eta', \vec{x}', \vec{k}, s)$$
$$= \frac{i2\pi^2}{k} u_i(\vec{k}, s) \partial^2 e^{i\vec{k}\cdot\vec{x}} \int_{\eta_i}^{\eta} d\eta' \frac{e^{-ik\eta'}}{\sqrt{2k}} \int_0^{\Delta\eta} dr r \sin(kr) \Big\{ \ln[\mu^2(\Delta\eta^2 - r^2)] - 1 \Big\}$$
$$= \frac{i2\pi^2}{k\sqrt{2k}} e^{i\vec{k}\cdot\vec{x}} u_i(\vec{k}, s) [-\partial_0^2 - k^2] \int_{\eta_i}^{\eta} d\eta' e^{-ik\eta'} \Delta\eta^2$$
$$\times \int_0^1 dz z \sin(\alpha z) \Big\{ \ln(1 - z^2) + 2\ln(\frac{\mu\alpha}{k}) - 1 \Big\}. \quad (332)$$

Here $\alpha \equiv k\Delta\eta$ and $\eta_i \equiv -1/H$ is the initial conformal time, corresponding to physical time $t = 0$. The integral over z is facilitated by the special function,

$$\xi(\alpha) \equiv \int_0^1 dz z \sin(\alpha z) \ln(1 - z^2) = \frac{2}{\alpha^2} \sin(\alpha) - \frac{1}{\alpha^2} \Big[\cos(\alpha) + \alpha \sin(\alpha) \Big]$$
$$\times \Big[\mathrm{si}(2\alpha) + \frac{\pi}{2} \Big] + \frac{1}{\alpha^2} \Big[\sin(\alpha) - \alpha\cos(\alpha) \Big] \Big[\mathrm{ci}(2\alpha) - \gamma - \ln(\frac{\alpha}{2}) \Big]. \quad (333)$$

Here γ is the Euler-Mascheroni constant and the sine and cosine integrals are,

$$\mathrm{si}(x) \equiv -\int_x^\infty dt \, \frac{\sin(t)}{t} = -\frac{\pi}{2} + \int_0^x dt \, \frac{\sin t}{t} \, , \quad (334)$$
$$\mathrm{ci}(x) \equiv -\int_x^\infty dt \, \frac{\cos t}{t} = \gamma + \ln(x) + \int_0^x dt \, \Big[\frac{\cos(t) - 1}{t} \Big] \, . \quad (335)$$

After substituting the ξ function and performing the elementary integrals, Equation 332 becomes,

$$\int d^4x' \Big\{ \frac{\ln(\mu^2 \Delta x_{++}^2)}{\Delta x_{++}^2} - \frac{\ln(\mu^2 \Delta x_{+-}^2)}{\Delta x_{+-}^2} \Big\} \Psi_i^0(\eta', \vec{x}', \vec{k}, s) = -\frac{i2\pi^2}{k\sqrt{2k}} e^{i\vec{k}\cdot\vec{x}} u_i(\vec{k}, s)$$
$$\times (\partial_{k\eta}^2 + 1) \int_{\eta_i}^{\eta} d\eta' e^{-ik\eta'} \Bigg\{ \alpha^2 \xi(\alpha) + \Big[2\ln(\frac{\mu\alpha}{k}) - 1 \Big] \Big[\sin(\alpha) - \alpha\cos(\alpha) \Big] \Bigg\}. \quad (336)$$

One can see that the integrand is of order $\alpha^3 \ln(\alpha)$ for small α, which

means we can pass the derivatives through the integral. After some rearrangements, the first key identity emerges,

$$\int d^4x' \Bigl\{ \frac{\ln(\mu^2 \Delta x^2_{++})}{\Delta x^2_{++}} - \frac{\ln(\mu^2 \Delta x^2_{+-})}{\Delta x^2_{+-}} \Bigr\} \Psi^0(\eta', \vec{x}'; \vec{k}, s)$$

$$= -i4\pi^2 k^{-1} \Psi^0(\eta, \vec{x}; \vec{k}, s) \int_{\eta_i}^{\eta} d\eta' e^{ik\Delta\eta} \Bigl\{ -\cos(k\Delta\eta) \int_0^{2k\Delta\eta} dt \, \frac{\sin(t)}{t}$$

$$+ \sin(k\Delta\eta) \Bigl[\int_0^{2k\Delta\eta} dt \, \Bigl(\frac{\cos(t)-1}{t} \Bigr) + 2\ln(2\mu\Delta\eta) \Bigr] \Bigr\}. \quad (337)$$

Note that we have written $e^{-ik\eta'} = e^{-ik\eta} \times e^{+ik\Delta\eta}$ and extracted the first phase to reconstruct the full tree order solution $\Psi^0(\eta, \vec{x}; \vec{k}, s) = \frac{e^{-ik\eta}}{\sqrt{2k}} u_i(\vec{k}, s) e^{i\vec{k}\cdot\vec{x}}$.

The second identity derives from acting a d'Alembertian on Equation 337. The d'Alembertian passes through the tree order solution to give,

$$\partial^2 \Psi^0(\eta, \vec{x}; \vec{k}, s) = \Psi^0(\eta, \vec{x}; \vec{k}, s) \partial_\eta (\partial_\eta - 2ik) \; . \quad (338)$$

Because the integrand goes like $\alpha \ln(\alpha)$ for small α, we can pass the first derivative through the integral to give,

$$\partial^2 \int d^4x' \Bigl\{ \frac{\ln(\mu^2 \Delta x^2_{++})}{\Delta x^2_{++}} - \frac{\ln(\mu^2 \Delta x^2_{+-})}{\Delta x^2_{+-}} \Bigr\} \Psi^0(\eta', \vec{x}'; \vec{k}, s)$$

$$= i4\pi^2 \Psi^0(\eta, \vec{x}; \vec{k}, s) \partial_\eta \int_{\eta_i}^{\eta} d\eta' \Bigl\{ \int_0^{2\alpha} dt \Bigl(\frac{e^{it}-1}{t} \Bigr) + 2\ln\Bigl(\frac{2\mu\alpha}{k}\Bigr) \Bigr\}. \quad (339)$$

We can pass the final derivative through the first integral but, for the second, we must carry out the integration. The result is our second key identity,

$$\partial^2 \int d^4x' \Bigl\{ \frac{\ln(\mu^2 \Delta x^2_{++})}{\Delta x^2_{++}} - \frac{\ln(\mu^2 \Delta x^2_{+-})}{\Delta x^2_{+-}} \Bigr\} \Psi^0(\eta', \vec{x}'; \vec{k}, s)$$

$$= i4\pi^2 \Psi^0(\eta, \vec{x}; \vec{k}, s) \Bigl\{ 2\ln\Bigl[\frac{2\mu}{H}(1+H\eta)\Bigr] + \int_{\eta_i}^{\eta} d\eta' \Bigl(\frac{e^{i2k\Delta\eta}-1}{\Delta\eta} \Bigr) \Bigr\}. \quad (340)$$

The final key identity is derived through the same procedures. Because they should be familiar by now we simply give the result,

$$\int d^4x' \Bigl\{ \frac{1}{\Delta x^2_{++}} - \frac{1}{\Delta x^2_{+-}} \Bigr\} \Psi^0(\eta', \vec{x}'; \vec{k}, s)$$

$$= -i4\pi^2 k^{-1} \Psi^0(\eta, \vec{x}; \vec{k}, s) \int_{\eta_i}^{\eta} d\eta' \, e^{ik\Delta\eta} \sin(k\Delta\eta) \; . \quad (341)$$

Table 19: Derivative operators U^I_{ij}: Their common prefactor is $\frac{\kappa^2 H^2}{2^8 \pi^4}$.

I	U^I_{ij}	I	U^I_{ij}
1	$(H^2 a a')^{-1}\ \partial\!\!\!/ \partial^4$	4	$-8\ \bar{\partial\!\!\!/} \partial^2$
2	$\frac{15}{2}\ \partial\!\!\!/ \partial^2$	5	$4\ \partial\!\!\!/ \nabla^2$
3	$-\ \bar{\partial\!\!\!/} \partial^2$	6	$7\ \partial\!\!\!/ \nabla^2$

5.2 Solving the Effective Dirac Equation

In this section we first evaluate the various nonlocal contributions using the three identities of the previous section. Then we evaluate the vastly simpler and, as it turns out, more important, local contributions. Finally, we solve for $\Psi^1(\eta, \vec{x}; \vec{k}, s)$ at late times.

The various nonlocal contributions to Equation 256 take the form,

$$\int d^4x' \sum_{I=1}^{5} U^I_{ij} \Big\{ \frac{\ln(\alpha_I^2 \Delta x^2_{++})}{\Delta x^2_{++}} - \frac{\ln(\alpha_I^2 \Delta x^2_{+-})}{\Delta x^2_{+-}} \Big\} \Psi^0_j(\eta', \vec{x}'; \vec{k}, s) \\ + \int d^4x' U^6_{ij} \Big\{ \frac{1}{\Delta x^2_{++}} - \frac{1}{\Delta x^2_{+-}} \Big\} \Psi^0_j(\eta', \vec{x}'; \vec{k}, s)\ . \quad (342)$$

The spinor differential operators U^I_{ij} are listed in Table 19. The constants α_I are μ for $I = 1, 2, 3$, and $\frac{1}{2}H$ for $I = 4, 5$.

As an example, consider the contribution from U^2_{ij}:

$$\frac{15}{2} \frac{\kappa^2 H^2}{2^8 \pi^4}\ \partial\!\!\!/ \partial^2 \int d^4x' \Big\{ \frac{\ln(\mu^2 \Delta x^2_{++})}{\Delta x^2_{++}} - \frac{\ln(\mu^2 \Delta x^2_{+-})}{\Delta x^2_{+-}} \Big\} \Psi^0(\eta', \vec{x}'; \vec{k}, s)$$

$$= \frac{15}{2} \frac{\kappa^2 H^2}{2^8 \pi^4}\ \partial\!\!\!/ \times i 4\pi^2 \Psi^0(\eta, \vec{x}; \vec{k}, s) \Big\{ 2 \ln\Big[\frac{2\mu}{H}(1 + H\eta)\Big] + \int_{\eta_i}^{\eta} d\eta' \Big(\frac{e^{2ik\Delta\eta} - 1}{\Delta\eta}\Big) \Big\}, \quad (343)$$

$$= \frac{\kappa^2 H^2}{2^6 \pi^2} i H \gamma^0 \Psi^0(\eta, \vec{x}; \vec{k}, s) \times \frac{15}{2} \frac{1}{1 + H\eta} \Big\{ e^{2i\frac{k}{H}(1 + H\eta)} + 1 \Big\}. \quad (344)$$

In these reductions we have used $i\ \partial\!\!\!/ \Psi^0(\eta, \vec{x}; \vec{k}, s) = i\gamma^0 \Psi^0(\eta, \vec{x}; \vec{k}, s)\, \partial_\eta$ and the second key identity 340. Recall from the Introduction that reliable predictions are only possible for late times, which corresponds to $\eta \to 0^-$. We therefore take this limit,

$$\frac{15}{2} \frac{\kappa^2 H^2}{2^8 \pi^4}\ \partial\!\!\!/ \partial^2 \int d^4x' \Big\{ \frac{\ln(\mu^2 \Delta x^2_{++})}{\Delta x^2_{++}} - \frac{\ln(\mu^2 \Delta x^2_{+-})}{\Delta x^2_{+-}} \Big\} \Psi^0(\eta', \vec{x}'; \vec{k}, s)$$

Table 20: Nonlocal contributions to $\int d^4x'[\Sigma](x;x')\Psi^0(\eta',\vec{x}';\vec{k},s)$ at late times. Multiply each term by $\frac{\kappa^2 H^2}{2^6\pi^2} \times iH\gamma^0\Psi^0(\eta,\vec{x};\vec{k},s)$.

I	Coefficient of the late time contribution from each U^I_{ij}
1	0
2	$\frac{15}{2}\Big\{\exp(2i\frac{k}{H})+1\Big\}$
3	$-i\frac{k}{H}\Big\{2\ln(\frac{2\mu}{H}) - \int_{\eta_i}^0 d\eta'\Big(\frac{\exp(-2ik\eta')-1}{\eta'}\Big)\Big\}$
4	$8i\frac{k}{H}\int_{\eta_i}^0 d\eta'\Big(\frac{\exp(-2ik\eta')-1}{\eta'}\Big)$
5	$4\frac{k^2}{H}\int_{\eta_i}^0 d\eta' e^{-2ik\eta'}\Big\{\int_0^{-2k\eta'} dt\Big(\frac{\exp(-it)-1}{t}\Big)+2\ln(H\eta')\Big\}$
6	$-\frac{7}{2}i\frac{k}{H}\Big\{\exp(2i\frac{k}{H})-1\Big\}$

$$\longrightarrow \frac{\kappa^2H^2}{2^6\pi^2} iH\gamma^0\Psi^0(\eta,\vec{x};\vec{k},s) \times \frac{15}{2}\Big\{\exp(2i\frac{k}{H})+1\Big\}. \tag{345}$$

The other five nonlocal terms have very similar reductions. Each of them also goes to $\frac{\kappa^2H^2}{2^6\pi^2} \times iH\gamma^0\Psi^0(\eta,\vec{x};\vec{k},s)$ times a finite constant at late times. We summarize the results in Table 20 and relegate the details to an appendix.

The next step is to evaluate the local contributions. This is a straightforward exercise in calculus, using only the properties of the tree order solution 281 and the fact that $\partial_\mu a = Ha^2\delta^0_\mu$. The result is,

$$\frac{i\kappa^2H^2}{2^6\pi^2}\int d^4x'\Big\{\frac{\ln(aa')}{H^2aa'}\,\partial\!\!\!/\partial^2 + \frac{15}{2}\ln(aa')\,\partial\!\!\!/ - 7\ln(aa')\bar{\partial\!\!\!/}\Big\}\delta^4(x-x')\Psi^0(\eta',\vec{x}';\vec{k},s)$$
$$= \frac{i\kappa^2H^2}{2^6\pi^2}\Big\{\frac{\ln(a)}{H^2a}\,\partial\!\!\!/\partial^2\Big(\frac{1}{a}\Psi^0(\eta,\vec{x};\vec{k},s)\Big) + \frac{1}{H^2a}\,\partial\!\!\!/\partial^2\Big(\frac{\ln(a)}{a}\Psi^0(\eta,\vec{x};\vec{k},s)\Big)$$
$$+\frac{15}{2}\Big(\ln(a)\,\partial\!\!\!/ + \partial\!\!\!/\ln(a)\Big)\Psi^0(\eta,\vec{x};\vec{k},s) - 14\ln(a)\,\bar{\partial\!\!\!/}\Psi^0(\eta,\vec{x};\vec{k},s)\Big\}, \tag{346}$$
$$= \frac{\kappa^2H^2}{2^6\pi^2} iH\gamma^0\Psi^0(\eta,\vec{x};\vec{k},s) \times \Big\{\frac{17}{2}a - 14i\frac{k}{H}\ln(a) - 2i\frac{k}{H}\Big\}. \tag{347}$$

The local quantum corrections 347 are evidently much stronger than their nonlocal counterparts in Table 20! Whereas the nonlocal terms approach

a constant, the leading local contribution grows like the inflationary scale factor, $a = e^{Ht}$. Even factors of $\ln(a)$ are negligible by comparison. We can therefore write the late time limit of the one loop field equation as,

$$i\not{\partial}\kappa^2\Psi^1(\eta,\vec{x};\vec{k},s) \longrightarrow \frac{\kappa^2H^2}{2^6\pi^2}\frac{17}{2}iHa\gamma^0\Psi^0(\eta,\vec{x};\vec{k},s) \; . \qquad (348)$$

The only way for the left hand side to reproduce such rapid growth is for the time derivative in $i\not{\partial}$ to act on a factor of $\ln(a)$,

$$i\gamma^\mu\partial_\mu \ln(a) = i\gamma^\mu\frac{Ha^2}{a}\delta^0_\mu = iHa\gamma^0 \; . \qquad (349)$$

We can therefore write the late time limit of the tree plus one loop mode functions as,

$$\Psi^0(\eta,\vec{x};\vec{k},s)+\kappa^2\Psi^1(\eta,\vec{x};\vec{k},s) \longrightarrow \Big\{1+\frac{\kappa^2H^2}{2^6\pi^2}\frac{17}{2}\ln(a)\Big\}\Psi^0(\eta,\vec{x};\vec{k},s) \; . \qquad (350)$$

All other corrections actually fall off at late times. For example, those from the $\ln(a)$ terms in Equation 347 go like $\ln(a)/a$.

There is a clear physical interpretation for the sort of solution we see in Equation 350. When the corrected field goes to the free field times a constant, that constant represents a field strength renormalization. When the quantum corrected field goes to the free field times a function of time that is independent of the form of the free field solution, it is natural to think in terms of a *time dependent field strength renormalization*,

$$\Psi(\eta,\vec{x};\vec{k},s) \longrightarrow \frac{\Psi^0(\eta,\vec{x};\vec{k},s)}{\sqrt{Z_2(t)}} \quad \text{where} \quad Z_2(t) = 1-\frac{17\kappa^2H^2}{2^6\pi^2}\ln(a)+O(\kappa^4) \; . \qquad (351)$$

Of course we only have the order κ^2 correction, so one does not know if this behavior persists at higher orders. If no higher loop correction supervenes, the field would switch from positive norm to negative norm at $\ln(a) = 2^6\pi^2/17\kappa^2H^2$. In any case, it is safe to conclude that perturbation theory must break down near this time.

5.3 Hartree Approximation

The appearance of a time-dependent field strength renormalization is such a surprising result that it is worth noting we can understand it on a simple,

qualitative level using the Hartree, or mean-field, approximation. This technique has proved useful in a wide variety of problems from atomic physics [83] and statistical mechanics [84], to nuclear physics [85] and quantum field theory [86]. Of particular relevance to our work is the insight the Hartree approximation provides into the generation of photon mass by inflationary particle production in SQED [87, 88, 89].

The idea is that we can approximate the dynamics of Fermi fields interacting with the graviton field operator, $h_{\mu\nu}$, by taking the expectation value of the Dirac Lagrangian in the graviton vacuum. To the order we shall need it, the Dirac Lagrangian is Equation 50,

$$\begin{aligned}\mathcal{L}_{\rm Dirac} = \overline{\Psi} i \not{\partial} \Psi + \frac{\kappa}{2}\Big\{ h \overline{\Psi} i \not{\partial} \Psi - h^{\mu\nu} \overline{\Psi} \gamma_{\mu} i \partial_{\nu} \Psi - h_{\mu\rho,\sigma} \overline{\Psi} \gamma^{\mu} J^{\rho\sigma} \Psi \Big\} \\ + \kappa^2 \Big[\frac{1}{8} h^2 - \frac{1}{4} h^{\rho\sigma} h_{\rho\sigma} \Big] \overline{\Psi} i \not{\partial} \Psi + \kappa^2 \Big[-\frac{1}{4} h h^{\mu\nu} + \frac{3}{8} h^{\mu\rho} h_{\rho}^{\ \nu} \Big] \overline{\Psi} \gamma_{\mu} i \partial_{\nu} \Psi \\ + \kappa^2 \Big[-\frac{1}{4} h h_{\mu\rho,\sigma} + \frac{1}{8} h^{\nu}_{\ \rho} h_{\nu\sigma,\mu} + \frac{1}{4} (h^{\nu}_{\ \mu} h_{\nu\rho})_{,\sigma} + \frac{1}{4} h^{\nu}_{\ \sigma} h_{\mu\rho,\nu} \Big] \overline{\Psi} \gamma^{\mu} J^{\rho\sigma} \Psi + O(\kappa^3). \end{aligned} \tag{352}$$

Of course the expectation value of a single graviton field is zero, but the expectation value of the product of two fields is the graviton propagator in Equation 74,

$$\begin{aligned}&\langle \Omega \mid T\Big[h_{\mu\nu}(x) h_{\rho\sigma}(x') \Big] \mid \Omega \rangle \\ &\qquad = i\Delta_A(x;x') \Big[{}_{\mu\nu} T^A_{\rho\sigma} \Big] + i\Delta_B(x;x') \Big[{}_{\mu\nu} T^B_{\rho\sigma} \Big] + i\Delta_C(x;x') \Big[{}_{\mu\nu} T^C_{\rho\sigma} \Big]. \end{aligned} \tag{353}$$

Recall the index factors from Equations 76-78,

$$\Big[{}_{\mu\nu} T^A_{\rho\sigma} \Big] = 2 \bar{\eta}_{\mu(\rho} \bar{\eta}_{\sigma)\nu} - \frac{2}{D-3} \bar{\eta}_{\mu\nu} \bar{\eta}_{\rho\sigma} \quad , \quad \Big[{}_{\mu\nu} T^B_{\rho\sigma} \Big] = -4 \delta^0_{\ (\mu} \bar{\eta}_{\nu)(\rho} \delta^0_{\sigma)} \ , \tag{354}$$

$$\Big[{}_{\mu\nu} T^C_{\rho\sigma} \Big] = \frac{2}{(D-2)(D-3)} \Big[(D-3) \delta^0_{\mu} \delta^0_{\nu} + \bar{\eta}_{\mu\nu} \Big] \Big[(D-3) \delta^0_{\rho} \delta^0_{\sigma} + \bar{\eta}_{\rho\sigma} \Big] \ . \tag{355}$$

Recall also that parenthesized indices are symmetrized and that a bar over a common tensor such as the Kronecker delta function denotes that its temporal components have been nulled,

$$\overline{\delta}^{\mu}_{\nu} \equiv \delta^{\mu}_{\nu} - \delta^{\mu}_{0} \delta^0_{\nu} \qquad , \qquad \overline{\eta}_{\mu\nu} \equiv \eta_{\mu\nu} + \delta^0_{\mu} \delta^0_{\nu} \ . \tag{356}$$

The three scalar propagators that appear in Equation 353 have complicated expressions 83-85 which imply the following results for their coincidence

limits and for the coincidence limits of their first derivatives,

$$\lim_{x'\to x} i\Delta_A(x;x') = \frac{H^{D-2}}{(4\pi)^{\frac{D}{2}}}\frac{\Gamma(D-1)}{\Gamma(\frac{D}{2})}\Big\{-\pi\cot\Big(\frac{\pi}{2}D\Big)+2\ln(a)\Big\}\,, \tag{357}$$

$$\lim_{x'\to x} \partial_\mu i\Delta_A(x;x') = \frac{H^{D-2}}{(4\pi)^{\frac{D}{2}}}\frac{\Gamma(D-1)}{\Gamma(\frac{D}{2})}\times Ha\delta^0_\mu = \lim_{x'\to x}\partial'_\mu i\Delta_A(x;x')\,, \tag{358}$$

$$\lim_{x'\to x} i\Delta_B(x;x') = \frac{H^{D-2}}{(4\pi)^{\frac{D}{2}}}\frac{\Gamma(D-1)}{\Gamma(\frac{D}{2})}\times -\frac{1}{D-2}\,, \tag{359}$$

$$\lim_{x'\to x} \partial_\mu i\Delta_B(x;x') = 0 = \lim_{x'\to x}\partial'_\mu i\Delta_B(x;x')\,, \tag{360}$$

$$\lim_{x'\to x} i\Delta_C(x;x') = \frac{H^{D-2}}{(4\pi)^{\frac{D}{2}}}\frac{\Gamma(D-1)}{\Gamma(\frac{D}{2})}\times \frac{1}{(D-2)(D-3)}\,, \tag{361}$$

$$\lim_{x'\to x} \partial_\mu i\Delta_C(x;x') = 0 = \lim_{x'\to x}\partial'_\mu i\Delta_C(x;x')\,. \tag{362}$$

We are interested in terms which grow at late times. Because the B-type and C-type propagators go to constants, and their derivatives vanish, they can be neglected. The same is true for the divergent constant in the coincidence limit of the A-type propagator. In the full theory it would be absorbed into a constant counterterm. Because the remaining, time dependent terms are finite, we may as well take $D=4$. Our Hartree approximation therefore amounts to making the following replacements in Equation 352,

$$h_{\mu\nu}h_{\rho\sigma} \longrightarrow \frac{H^2}{4\pi^2}\ln(a)\Big[\overline{\eta}_{\mu\rho}\overline{\eta}_{\nu\sigma}+\overline{\eta}_{\mu\sigma}\overline{\eta}_{\nu\rho}-2\overline{\eta}_{\mu\nu}\overline{\eta}_{\rho\sigma}\Big]\,, \tag{363}$$

$$h_{\mu\nu}h_{\rho\sigma,\alpha} \longrightarrow \frac{H^2}{8\pi^2}Ha\delta^0_\alpha\Big[\overline{\eta}_{\mu\rho}\overline{\eta}_{\nu\sigma}+\overline{\eta}_{\mu\sigma}\overline{\eta}_{\nu\rho}-2\overline{\eta}_{\mu\nu}\overline{\eta}_{\rho\sigma}\Big]\,. \tag{364}$$

It is now just a matter of contracting Equations 363-364 appropriately to produce each of the quadratic terms in Equation 352. For example, the first term gives,

$$\frac{\kappa^2}{8}h^2\overline{\Psi}i\partial\!\!\!/\Psi \longrightarrow \frac{\kappa^2H^2}{2^5\pi^2}\ln(a)\Big[\eta^{\mu\nu}\eta^{\rho\sigma}\Big]\Big[\bar{\eta}_{\mu\rho}\bar{\eta}_{\nu\sigma}+\bar{\eta}_{\mu\sigma}\bar{\eta}_{\nu\rho}-2\bar{\eta}_{\mu\nu}\bar{\eta}_{\rho\sigma}\Big]\overline{\Psi}i\partial\!\!\!/\Psi, \tag{365}$$

$$= \frac{\kappa^2H^2}{2^5\pi^2}\ln(a)\Big[3+3-18\Big]\overline{\Psi}i\partial\!\!\!/\Psi\,. \tag{366}$$

The second quadratic term gives a proportional result,

$$\frac{-\kappa^2}{4}h^{\rho\sigma}h_{\rho\sigma}\overline{\Psi}i\partial\!\!\!/\Psi \longrightarrow \frac{-\kappa^2H^2}{2^4\pi^2}\ln(a)[9+3-6\Big]\overline{\Psi}i\partial\!\!\!/\Psi\,. \tag{367}$$

The total for these first two terms is $\frac{-3\kappa^2 H^2}{4\pi^2} \ln(a) \overline{\Psi} i \partial\!\!\!/ \Psi$.

The third and fourth of the quadratic terms in Equation 352 result in only spatial derivatives,

$$\frac{-\kappa^2}{4} h h^{\mu\nu} \overline{\Psi} \gamma_\mu i \partial_\nu \Psi \longrightarrow \frac{-\kappa^2 H^2}{2^4 \pi^2} \ln(a) \Big[1 + 1 - 6 \Big] \overline{\Psi} i \bar{\partial\!\!\!/} \Psi \; , \qquad (368)$$

$$\frac{3}{8} \kappa^2 h^{\mu\rho} h^\nu_{\rho} \overline{\Psi} \gamma_\mu i \partial_\nu \Psi \longrightarrow \frac{3\kappa^2 H^2}{2^5 \pi^2} \ln(a) \Big[3 + 1 - 2 \Big] \overline{\Psi} i \bar{\partial\!\!\!/} \Psi \; . \qquad (369)$$

The total for this type of contribution is $\frac{7\kappa^2 H^2}{2^4 \pi^2} \ln(a) \overline{\Psi} i \bar{\partial\!\!\!/} \Psi$.

The final four quadratic terms in Equation 352 involve derivatives acting on at least one of the two graviton fields,

$$- \frac{\kappa^2}{4} h h_{\mu\rho,\sigma} \overline{\Psi} \gamma^\mu J^{\rho\sigma} \Psi \longrightarrow \frac{-\kappa^2 H^2}{2^5 \pi^2} H a \Big[1 + 1 - 6 \Big] \overline{\eta}_{\mu\rho} \overline{\Psi} \gamma^\mu J^{\rho 0} \Psi \; , \qquad (370)$$

$$\frac{\kappa^2}{8} h^\nu_{\rho} h_{\nu\sigma,\mu} \overline{\Psi} \gamma^\mu J^{\rho\sigma} \Psi \longrightarrow \frac{\kappa^2 H^2}{2^6 \pi^2} H a \Big[3 + 1 - 2 \Big] \overline{\eta}_{\rho\sigma} \overline{\Psi} \gamma^0 J^{\rho\sigma} \Psi \; , \qquad (371)$$

$$\frac{\kappa^2}{4} \Big(h^\nu_{\mu} h_{\nu\rho} \Big)_{,\sigma} \overline{\Psi} \gamma^\mu J^{\rho\sigma} \Psi \longrightarrow \frac{\kappa^2 H^2}{2^4 \pi^2} H a \Big[3 + 1 - 2 \Big] \overline{\eta}_{\mu\rho} \overline{\Psi} \gamma^\mu J^{\rho 0} \Psi \; , \qquad (372)$$

$$\frac{\kappa^2}{4} h^\nu_{\sigma} h_{\mu\rho,\nu} \overline{\Psi} \gamma^\mu J^{\rho\sigma} \Psi \longrightarrow 0 \; . \qquad (373)$$

The second of these contributions vanishes owing to the antisymmetry of the Lorentz representation matrices, $J^{\mu\nu} \equiv \frac{i}{4}[\gamma^\mu, \gamma^\nu]$, whereas $\overline{\eta}_{\mu\rho} \gamma^\mu J^{\rho 0} = -\frac{3i}{2} \gamma^0$. Hence the sum of all four terms is $\frac{-3\kappa^2 H^2}{8\pi^2} H a \overline{\Psi} i \gamma^0 \Psi$.

Combining these results gives,

$$\Big\langle \mathcal{L}_{\rm Dirac} \Big\rangle = \overline{\Psi} i \partial\!\!\!/ \Psi - \frac{3\kappa^2 H^2}{4\pi^2} \ln(a) \overline{\Psi} i \partial\!\!\!/ \Psi - \frac{3\kappa^2 H^2}{8\pi^2} H a \overline{\Psi} i \gamma^0 \Psi + \frac{7\kappa^2 H^2}{16\pi^2} \ln(a) \overline{\Psi} i \bar{\partial\!\!\!/} \Psi + O(\kappa^4), \qquad (374)$$

$$= \overline{\Psi} \Big[1 - \frac{3\kappa^2 H^2}{8\pi^2} \ln(a) \Big] i \partial\!\!\!/ \Big[1 - \frac{3\kappa^2 H^2}{8\pi^2} \ln(a) \Big] \Psi + \frac{7\kappa^2 H^2}{16\pi^2} \ln(a) \overline{\Psi} i \bar{\partial\!\!\!/} \Psi + O(\kappa^4). \qquad (375)$$

If we express the equations associated with Equation 375 according to the perturbative scheme of Section 2, the first order equation is,

$$i \partial\!\!\!/ \kappa^2 \Psi^1(\eta, \vec{x}; \vec{k}, s) = \frac{\kappa^2 H^2}{2^6 \pi^2} i H \gamma^0 \Psi^0(\eta, \vec{x}; \vec{k}, s) \Big\{ 24 a - 28 i \frac{k}{H} \ln(a) \Big\} \; . \qquad (376)$$

This is similar, but not identical to, what we got in expression 347 from the delta function terms of the actual one loop self-energy in Equation 251.

In particular, the exact calculation gives $\frac{17}{2}a - 14i\frac{k}{H}\ln(a)$, rather than the Hartree approximation of $24a - 28i\frac{k}{H}\ln(a)$. Of course the $\ln(a)$ terms make corrections to Ψ^1 which fall like $\ln(a)/a$, so the real disagreement between the two methods is limited to the differing factors of $\frac{17}{2}$ versus 24.

We are pleased that such a simple technique comes so close to recovering the result of a long and tedious calculation. The slight discrepancy is no doubt due to terms in the Dirac Lagrangian by Equation 352 which are linear in the graviton field operator. As described in relation 289 of section 2, the linearized effective field $\Psi_i(x;\vec{k},s)$ represents $a^{\frac{D-1}{2}}$ times the expectation value of the anti-commutator of the Heisenberg field operator $\psi_i(x)$ with the free fermion creation operator $b(\vec{k},s)$. At the order we are working, quantum corrections to $\Psi_i(x;\vec{k},s)$ derive from perturbative corrections to $\psi_i(x)$ which are quadratic in the free graviton creation and annihilation operators. Some of these corrections come from a single $hh\overline{\psi}\psi$ vertex, while others derive from two $h\overline{\psi}\psi$ vertices. The Hartree approximation recovers corrections of the first kind, but not the second, which is why we believe it fails to agree with the exact result. Yukawa theory presents a fully worked-out example [11, 12, 90] in which the *entire* lowest-order correction to the fermion mode functions derives from the product of two such linear terms, so the Hartree approximation fails completely in that case.

6 CONCLUSIONS

We have used dimensional regularization to compute quantum gravitational corrections to the fermion self-energy at one loop order in a locally de Sitter background. Our regulated result is Equation 238. Although Dirac + Einstein is not perturbatively renormalizable [18] we obtained a finite result shown by Equation 251 by absorbing the divergences with BPHZ counterterms.

For this 1PI function, and at one loop order, only three counterterms are necessary. None of them represents redefinitions of terms in the Lagrangian of Dirac + Einstein. Two of the required counterterms of Equation 92 are generally coordinate invariant fermion bilinears of dimension six. The third counterterm of Equation 104 is the only other fermion bilinear of dimension six which respects the symmetries shown by Equations 61-66 of our de Sitter noninvariant gauge shown in Equation 60 and also obeys the reflection property shown in Equation 103 of the self-energy for massless fermions.

Although parts of this computation are quite intricate we have good confidence that Equation 251 is correct for three reasons. First, there is the flat space limit of taking H to zero while taking the conformal time to be $\eta = -e^{-Ht}/H$ with t held fixed. This checks the leading conformal contributions. Our second reason for confidence is the fact that all divergences can be absorbed using just the three counterterms we have inferred in chapter 2 on the basis of symmetry. This was by no means the case for individual terms; many separate pieces must be added to eliminate other divergences. The final check comes from the fact that the self-energy of a massless fermion must be odd under interchange of its two coordinates. This was again not true for separate contributions, yet it emerged when terms were summed.

Although our renormalized result could be changed by altering the finite parts of the three BPHZ counterterms, this does not affect its leading behavior in the far infrared. It is simple to be quantitative about this. Were we to make finite shifts $\Delta\alpha_i$ in our counterterms Equation 247 the induced change in the renormalized self-energy would be,

$$-i\Big[\Delta\Sigma^{\rm ren}\Big](x;x') = -\kappa^2 \Bigg\{ \frac{\Delta\alpha_1}{aa'} \not{\partial}\partial^2 + 12\Delta\alpha_2 H^2 \not{\partial} + \Delta\alpha_3 H^2 \overline{\not{\partial}} \Bigg\} \delta^4(x-x') \; . \quad (377)$$

No physical principle seems to fix the $\Delta\alpha_i$ so any result that derives from their values is arbitrary. This is why BPHZ renormalization does not yield a complete theory. However, at late times (which accesses the far infrared because all momenta are redshifted by $a(t) = e^{Ht}$) the local part of the renormalized self-energy of Equation 251 is dominated by the large logarithms,

$$\frac{\kappa^2}{2^6\pi^2} \Bigg\{ \frac{\ln(aa')}{aa'} \not{\partial}\partial^2 + \frac{15}{2} \ln(aa') H^2 \not{\partial} - 7 \ln(aa') H^2 \overline{\not{\partial}} \Bigg\} \delta^4(x-x') \; . \qquad (378)$$

The coefficients of these logarithms are finite and completely fixed by our calculation. As long as the shifts $\Delta\alpha_i$ are finite, their impact Equation 377 must eventually be dwarfed by the large logarithms in Equation 378.

None of this should seem surprising, although it does with disturbing regularity. The comparison we have just made is a standard feature of low energy effective field theory and has a very old and distinguished pedigree [23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37]. Loops of massless particles make finite, nonanalytic contributions which cannot be changed by local counterterms and which dominate the far infrared. Further, these effects must occur as well, with precisely the same numerical values, in whatever

fundamental theory ultimately resolves the ultraviolet problem of quantum gravity. That is why Feinberg and Sucher got exactly the same long range force from the exchange of massless neutrinos using Fermi theory [25, 26] as one would get from the Standard Model [26].

So we can use Equation 251 reliably in the far infrared. Our motivation for undertaking this exercise was to search for a gravitational analogue of what Yukawa-coupling a massless, minimally coupled scalar does to massless fermions during inflation [11]. Obtaining Equation 251 completes the first part in that program. In the second stage we used the Schwinger-Keldysh formalism to include one loop, quantum gravitational corrections to the Dirac equation. Because Dirac + Einstein is not perturbatively renormalizable, it makes no sense to solve this equation generally. However, the equation should give reliable predictions at late times when the arbitrary finite parts of the BPHZ counterterms Equation 246 are insignificant compared to the completely determined factors of $\ln(aa')$ on terms of Equations 258-260 which otherwise have the same structure. In this late time limit we find that the one loop corrected, spatial plane wave mode functions behave as if the tree order mode functions were simply subject to a time-dependent field strength renormalization,

$$Z_2(t) = 1 - \frac{17}{4\pi} GH^2 \ln(a) + O(G^2) \quad \text{where} \quad G = 16\pi\kappa^2 \; . \tag{379}$$

If unchecked by higher loop effects, this would vanish at $\ln(a) \simeq 1/GH^2$. What actually happens depends upon higher order corrections, but there is no way to avoid perturbation theory breaking down at this time, at least in this gauge.

Might this result be a gauge artifact? One reaches different gauges by making field dependent transformations of the Heisenberg operators. We have worked out the change in Equation 325 this induces in the linearized effective field, but the result is not simple. Although the linearized effective field obviously changes when different gauge conditions are employed to compute it, we believe (but have not proven) that the late time factors of $\ln(a)$ do not change.

It is important to realize that the 1PI functions of a gauge theory in a fixed gauge are not devoid of physical content by virtue of depending upon the gauge. In fact, they encapsulate the physics of a quantum gauge field every bit as completely as they do when no gauge symmetry is present. One extracts this physics by forming the 1PI functions into gauge independent

and physically meaningful combinations. The S-matrix accomplishes this in flat space quantum field theory. Unfortunately, the S-matrix fails to exist for Dirac + Einstein in de Sitter background, nor would it correspond to an experiment that could be performed if it did exist [91, 92, 93].

If it is conceded that we know what it means to release the universe in a free state then it would be simple enough — albeit tedious — to construct an analogue of $\psi_i(x)$ which is invariant under gauge transformations that do not affect the initial value surface. For example, one might extend to fermions the treatment given for pure gravity by [94]:

- Propagate an operator-valued geodesic a fixed invariant time from the initial value surface;
- Use the spin connection $A_{\mu cd} J^{cd}$ to parallel transport along the geodesic; and
- Evaluate ψ at the operator-valued geodesic, in the Lorentz frame which is transported from the initial value surface.

This would make an invariant, as would any number of other constructions [95]. For that matter, the gauge-fixed 1PI functions also correspond to the expectation values of invariant operators [82]. Mere invariance does not guarantee physical significance, nor does gauge dependence preclude it.

What is needed is for the community to agree upon a relatively simple set of operators which stand for experiments that could be performed in de Sitter space. There is every reason to expect a successful outcome because the last few years have witnessed a resolution of the similar issue of how to measure quantum gravitational back-reaction during inflation, driven either by a scalar inflaton [96, 97, 98, 99] or by a bare cosmological constant [100]. That process has begun for quantum field theory in de Sitter space [91, 92, 95, 100] and one must wait for it to run its course. In the meantime, it is safest to stick with what we have actually shown: perturbation theory must break down for Dirac + Einstein in the simplest gauge.

This is a surprising result but we were able to understand it qualitatively using the Hartree approximation in which one takes the expectation value of the Dirac Lagrangian in the graviton vacuum. The physical interpretation seems to be that fermions propagate through an effective geometry whose ever-increasing deviation from de Sitter is controlled by inflationary graviton production. At one loop order the fermions are passive spectators to this effective geometry.

It is significant that inflationary graviton production enhances fermion mode functions by a factor of $\ln(a)$ at one loop. Similar factors of $\ln(a)$ have been found in the graviton vacuum energy [65, 66]. These infrared logarithms also occur in the vacuum energy and mode functions of a massless, minimally coupled scalar with a quartic self-interaction [56, 57, 101], and in the VEV's of almost all operators in Yukawa theory [90] and SQED [102, 103]. A recent all orders analysis was not even able to exclude the possibility that they might contaminate the power spectrum of primordial density fluctuations [104, 105, 106]!

The fact that infrared logarithms grow without bound raises the exciting possibility that quantum gravitational corrections may be significant during inflation, in spite of the minuscule coupling constant of $GH^2 \lesssim 10^{-12}$. However, the only thing one can legitimately conclude from the perturbative analysis is that infrared logarithms cause perturbation theory to break down, in our gauge, if inflation lasts long enough. Inferring what happens after this breakdown requires a nonperturbative technique.

Starobinskiĭ has long advocated that a simple stochastic formulation of scalar potential models serves to reproduce the leading infrared logarithms of these models at each order in perturbation theory [107]. This fact has recently been proved to all orders [108, 109]. When the scalar potential is bounded below it is even possible to sum the series of leading infrared logarithms and infer their net effect at asymptotically late times [110]! Applying Starobinskiĭ's technique to more complicated theories which also show infrared logarithms is a formidable problem, but solutions have recently been obtained for Yukawa theory [90] and for SQED [103]. It would be very interesting to see what this technique gives for the infrared logarithms we have exhibited, to lowest order, in Dirac + Einstein. And it should be noted that even the potentially complicated, invariant operators which might be required to settle the gauge issue would be straightforward to compute in such a stochastic formulation.

7 NONLOCAL TERMS FROM TABLE 5.2

It is important to establish that the nonlocal terms make no significant contribution at late times, so we will derive the results summarized in Table 20. For simplicity we denote as $[U^I]$ the contribution from each operator U^I_{ij} in Table 19. We also abbreviate $\Psi^0(\eta, \vec{x}; \vec{k}, s)$ as $\Psi^0(x)$.

Owing to the factor of $1/a'$ in U^1_{ij}, and to the larger number of derivatives, the reduction of $[U^1]$ is atypical,

$$[\mathrm{U}^1] \equiv \frac{\kappa^2}{2^8\pi^4}\frac{1}{a}\,\bar{\partial}\!\!\!/\,\partial^4 \int d^4x' \frac{1}{a'}\Bigg\{\frac{\ln(\mu^2\Delta x^2_{++})}{\Delta x^2_{++}} - \frac{\ln(\mu^2\Delta x^2_{+-})}{\Delta x^2_{+-}}\Bigg\}\Psi^0(x')\ , \qquad (380)$$

$$= \frac{-i\kappa^2}{2^6\pi^2 a}\gamma^0\Psi^0(x)\Big[-2ik\partial_\eta + \partial^2_\eta\Big]\Bigg\{\partial_\eta \int_{\eta_i}^{\eta} d\eta'(-H\eta')\Big(\frac{e^{2ik\Delta\eta}-1}{\Delta\eta}\Big) + \partial^2_\eta \int_{\eta_i}^{\eta} d\eta'(-2H\eta')\ln(2\mu\Delta\eta)\Bigg\}, \quad (381)$$

$$= \frac{-i\kappa^2}{2^6\pi^2 a}\gamma^0\Psi^0\Big(-2ik + \partial_\eta\Big)\Bigg\{-\frac{e^{2ik(\eta+\frac{1}{H})}-1}{(\eta+\frac{1}{H})^2} + \frac{(2ik-H)e^{2ik(\eta+\frac{1}{H})}}{\eta+\frac{1}{H}} - \frac{3H^2}{(1+H\eta)} + \frac{2H^3\eta}{(1+H\eta)^2}\Bigg\}\ , \qquad (382)$$

$$= \frac{\kappa^2 H^2}{2^6\pi^2}(H\eta)iH\gamma^0\Psi\Bigg\{\frac{2\Big[e^{\frac{2ik}{H}(1+H\eta)} - 1 - 2H\eta\Big]}{(1+H\eta)^3} + \frac{(1-\frac{2ik}{H})e^{\frac{2ik}{H}(1+H\eta)}}{(1+H\eta)^2} + \frac{5 - 4ik\eta - \frac{2ik}{H}}{(1+H\eta)^2} + \frac{\frac{6ik}{H}}{1+H\eta}\Bigg\}\ . \quad (383)$$

This expression actually vanishes in the late time limit of $\eta \to 0^-$.

$[U^2]$ was reduced in Section 4 so we continue with $[U^3]$,

$$[U^3] \equiv -\frac{\kappa^2 H^2}{2^8\pi^4}\,\bar{\partial}\!\!\!/\,\partial^2\!\int d^4x'\Bigg\{\frac{\ln(\mu^2\Delta x^2_{++})}{\Delta x^2_{++}} - \frac{\ln(\mu^2\Delta x^2_{+-})}{\Delta x^2_{+-}}\Bigg\}\Psi^0(x')\ , \qquad (384)$$

$$= -\frac{\kappa^2 H^2}{2^8\pi^4}\bar{\partial}\!\!\!/\, i4\pi^2\Psi^0(x)\Bigg\{2\ln\Big[\frac{2\mu}{H}(1+H\eta)\Big] + \int_{\eta_i}^{\eta} d\eta'\Big(\frac{e^{2ik\Delta\eta}-1}{\Delta\eta}\Big)\Bigg\}, \quad (385)$$

$$= \frac{\kappa^2 H^2}{2^6\pi^2}k\gamma^0\Psi^0(x)\Bigg\{2\ln\Big[\frac{2\mu}{H}(1+H\eta)\Big] + \int_{\eta_i}^{\eta} d\eta'\Big(\frac{e^{2ik\Delta\eta}-1}{\Delta\eta}\Big)\Bigg\}\ , \qquad (386)$$

$$\longrightarrow \frac{\kappa^2 H^2}{2^6\pi^2}iH\gamma^0\Psi^0(x) \times -\frac{ik}{H}\Bigg\{2\ln(\frac{2\mu}{H}) - \int_{\eta_i}^{0} d\eta'\Big(\frac{e^{-2ik\eta'}-1}{\eta'}\Big)\Bigg\}. \qquad (387)$$

U^4_{ij} has the same derivative structure as U^3_{ij}, so $[U^4]$ follows from Equation 387,

$$[U^4] \equiv -\frac{\kappa^2 H^2}{2^8\pi^4} \times 8\,\bar{\partial}\!\!\!/\,\partial^2\!\int d^4x'\Bigg\{\frac{\ln(\frac{1}{4}H^2\Delta x^2_{++})}{\Delta x^2_{++}} - \frac{\ln(\frac{1}{4}H^2\Delta x^2_{+-})}{\Delta x^2_{+-}}\Bigg\}\Psi^0(x')\ , \quad (388)$$

$$= \frac{\kappa^2 H^2}{2^6 \pi^2} 8k\gamma^0 \Psi^0(x) \Biggl\{ 2\ln\Bigl[(1+H\eta)\Bigr] + \int_{\eta_i}^{\eta} d\eta' \Bigl(\frac{e^{2ik\Delta\eta}-1}{\Delta\eta} \Bigr) \Biggr\} , \quad (389)$$

$$\longrightarrow \frac{\kappa^2 H^2}{2^6 \pi^2} iH\gamma^0 \Psi^0(x) \times 8i\frac{k}{H} \int_{\eta_i}^{0} d\eta' \Bigl(\frac{e^{-2ik\eta'}-1}{\eta'} \Bigr) . \quad (390)$$

U^5_{ij} has a Laplacian rather than a d'Alembertian so we use identity 337 for $[U^5]$. We also employ the abbreviation $k\Delta\eta = \alpha$,

$$[U^5] \equiv 4\frac{\kappa^2 H^2}{2^8 \pi^4} \partial\!\!\!/ \nabla^2 \!\int d^4x' \Biggl\{ \frac{\ln(\mu^2 \Delta x^2_{++})}{\Delta x^2_{++}} - \frac{\ln(\mu^2 \Delta x^2_{+-})}{\Delta x^2_{+-}} \Biggr\} \Psi^0(x') , \quad (391)$$

$$= 4\frac{\kappa^2 H^2}{2^8 \pi^4} \partial\!\!\!/ \nabla^2 \Bigl(\frac{-4i\pi^2}{k} \Bigr) \Psi^0(x) \int_{\eta_i}^{\eta} d\eta' e^{i\alpha}$$
$$\times \Biggl\{ -\cos(\alpha) \int_0^{2\alpha} dt \, \frac{\sin(t)}{t} + \sin(\alpha) \Bigl[\int_0^{2\alpha} dt \Bigl(\frac{\cos(t)-1}{t} \Bigr) + 2\ln\Bigl(\frac{H\alpha}{k} \Bigr) \Bigr] \Biggr\}, \quad (392)$$

$$= \frac{\kappa^2 H^2}{2^6 \pi^2} iH\gamma^0 \Psi^0(x) \times 4\frac{k^2}{H} \int_{\eta_i}^{\eta} d\eta' e^{2i\alpha} \Bigl[\int_0^{2\alpha} dt \Bigl(\frac{e^{-it}-1}{t} \Bigr) + \ln(H\Delta\eta)^2 \Bigr] , \quad (393)$$

$$\longrightarrow \frac{\kappa^2 H^2}{2^6 \pi^2} iH\gamma^0 \Psi^0(x) \times 4\frac{k^2}{H} \int_{\eta_i}^{0} d\eta' e^{2i\alpha} \Bigl[\int_0^{2\alpha} dt \Bigl(\frac{e^{-it}-1}{t} \Bigr) + \ln(H\eta')^2 \Bigr]. \quad (394)$$

U^6_{ij} has the same derivative structure as U^5_{ij} but it acts on a different integrand. We therefore apply identity 341 for $[U^6]$,

$$[U^6] \equiv 7\frac{\kappa^2 H^2}{2^8 \pi^4} \partial\!\!\!/ \nabla^2 \!\int d^4x' \Biggl\{ \frac{1}{\Delta x^2_{++}} - \frac{1}{\Delta x^2_{+-}} \Biggr\} \Psi^0(x') , \quad (395)$$

$$= 7\frac{\kappa^2 H^2}{2^8 \pi^4} \partial\!\!\!/ \nabla^2 \times (-i4\pi^2) k^{-1} \Psi^0(x) \int_{\eta_i}^{\eta} d\eta' e^{ik\Delta\eta} \sin(k\Delta\eta) , \quad (396)$$

$$= \frac{\kappa^2 H^2}{2^6 \pi^2} iH\gamma^0 \Psi^0(x) \times -\frac{7}{2} \frac{ik}{H} \Bigl[e^{\frac{2ik}{H}(1+H\eta)} - 1 \Bigr] , \quad (397)$$

$$\longrightarrow \frac{\kappa^2 H^2}{2^6 \pi^2} iH\gamma^0 \Psi^0(x) \times -\frac{7}{2} \frac{ik}{H} \Bigl[e^{\frac{2ik}{H}} - 1 \Bigr] . \quad (398)$$

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BIOGRAPHICAL SKETCH

Shun-Pei Miao came from Taiwan. She took her undergraduate degree in physics at National Taiwan Normal University (NTNU) in 1997. After that, she got a teaching job in a senior high school. Two years later she went back to school and in 2001 took a master's degree under the direction of Professor Pei-Ming Ho at National Taiwan University (NTU). Her research led to a published paper entitled, "Noncommutative Differential Calculus for D-Brane in Nonconstant B-Field Background," Phys. Rev, **D64**: 126002, 2001, hep-th/0105191. After completing her master's degree, she was fortunate to get a job at National Taiwan Normal University (NTNU) and she planned to study abroad.

Miao came to the UF in the fall of 2002 and passed the Preliminary Exam on her first attempt. She passed the graduate core courses during her first year. In 2003-4 she took particle physics, quantum field theory and Professor Fry's cosmology special topics course. She took general relativity in 2004-5. In 2005-6 she took the standard model. In the fall of 2006 she won a Marie-Curie Fellowship to attend a trimester at the Institute of Henri Poincaré entitled, "Gravitational Waves, Relativistic Astrophysics and Cosmology." In the spring of 2007 she took Professor Sikivie's dark matter course.

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